Higher order partial derivatives

Def: $\mathcal{U} \subset \mathbb{R}^n$ open, $f : \mathcal{U} \longrightarrow \mathbb{R}$, $a \in \mathcal{U}$

We set $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(a) := \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)(a)$

whenever it makes sense.

"second-order partial derivative"

Comment: "whenever it makes sense" means that $\dfrac{\partial f}{\partial x_i}$ exists in a small ball around a and admits a directional derivative at a along e_j.

Comment: we first differentiate with respect to x_i and then with respect to x_j (we read from right to left)

More generally, we set $\dfrac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}}(a) := \dfrac{\partial}{\partial x_{i_k}}\left(\dfrac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\left(\dfrac{\partial f}{\partial x_{i_1}}\right)\right)\right)(a)$

whenever it makes sense.

"partial derivative of order k"

Other notation: $\partial_{x_{i_k}} \partial_{x_{i_{k-1}}} \cdots \partial_{x_{i_1}} f$

Comment: Again we read from right to left: we first differentiate w.r.t x_{i_1} then x_{i_2}, ..., then x_{i_k}

Def: f is of class C^k if all its partial derivatives up to order k exist and are continuous ← don't forget the continuity | $C^0 \equiv$ continuous ⚠

$C^1 \equiv$ "continuously differentiable"

Ex : the order matters :

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y\dfrac{x^2-y^2}{x^2+y^2} + \dfrac{4x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

indeed $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{t\to 0} \dfrac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \dfrac{0}{t} = 0$

Similarly :

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} x\dfrac{x^2-y^2}{x^2+y^2} - \dfrac{4x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \frac{\partial f}{\partial y}(0,0)}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(t,0) - \frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t\to 0} \frac{-t}{t} = -1$$

Hence $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$

Theorem : C^k functions are closed by the elementary operations

↳ Before the example

Nevertheless, we have the following result

Theorem: (Clairaut [1740], Schwarz [First correct proof 1873]) In MAT237, we use "Clairaut's thm"

$\mathcal{U} \subset \mathbb{R}^m$, $f: \mathcal{U} \to \mathbb{R}$ of class C^2 on $\mathcal{U}$ (Δ), $a \in \mathcal{U}$

Then $\forall i,j = 1, \dots, m$, $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(a) = \dfrac{\partial^2 f}{\partial x_j \partial x_i}(a)$

"If the second partial derivatives are continuous then the order doesn't matter"

Δ WLOG: we assume $\mathcal{U} \subset \mathbb{R}^2$, $a = (x_0, y_0) \in \mathcal{U}$

WTS: $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$

Let $h > 0$, $k > 0$ s.t. $[x_0, x_0+h] \times [y_0, y_0+k] \subset \mathcal{U}$

Set $\delta_{h,k} = f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)$

Define $\varphi: [x_0, x_0+h] \to \mathbb{R}$ by $\varphi(x) = f(x, y_0+k) - f(x, y_0)$

then $\delta_{h,k} = \varphi(x_0+h) - \varphi(x_0)$

MVT to φ: $\exists \theta_1 \in (0,1)$ s.t. $\delta_{h,k} = \varphi(x_0+h) - \varphi(x_0) = h\varphi'(x_0+\theta_1 h)$

ie: $\delta_{h,k} = h\left(\dfrac{\partial f}{\partial x}(x_0+\theta_1 h, y_0+k) - \dfrac{\partial f}{\partial x}(x_0+\theta_1 h, y_0)\right)$

$\psi: [y_0, y_0+k] \to \mathbb{R}$ defined by $\psi(y) = \dfrac{\partial f}{\partial x}(x_0+\theta_1 h, y)$

By the MVT to ψ, $\exists \theta_2 \in (0,1)$ s.t.

$$\delta_{h,k} = hk \frac{\partial^2 f}{\partial y \partial x}(x_0+\theta_1 h, y_0+\theta_2 k)$$

Similarly, by repeating the above with y then x, $\exists \theta_3, \theta_4 \in (0,1)$

s.t. $\delta_{h,k} = hk \dfrac{\partial^2 f}{\partial x \partial y}(x_0+\theta_3 h, y_0+\theta_4 k)$

Therefore: $\dfrac{\partial^2 f}{\partial x \partial y}(x_0+\theta_3 h, y_0+\theta_4 k) = \dfrac{\partial^2 f}{\partial y \partial x}(x_0+\theta_1 h, y_0+\theta_2 k)$

$$\downarrow (h,k)\to(0,0) \qquad\qquad \downarrow (h,k)\to(0,0)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

since $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are continuous

$\square$

By an induction, we get that

Corollary: $U \subset \mathbb{R}^m$ open, $f: U \to \mathbb{R}$ of class C^k, $a \in U$

Then $\dfrac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}}(a)$ doesn't depend on the order of the $i_1, \dots, i_k$

Notation: if f is of class C^k, since the order doesn't matter, the following notation is quite useful:

$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$$

$$\partial^\alpha f(a) = \frac{\partial^{\alpha_1 + \dots + \alpha_m} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}}(a)$$

Homework: do the examples and questions of §2.5 of the lecture notes.