University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

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Question 1 (At home)**.** *Review the in-class questions from Sep 17 and from Sep. 24 (the solutions are online).*

Question 2. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$.*

1. *Prove that if A and B are open then $A \times B \subset \mathbb{R}^{n+m}$ is too.*
2. *Prove that if A and B are closed then $A \times B \subset \mathbb{R}^{n+m}$ is too.*

Question 3. *Let $A, B \subset \mathbb{R}^n$. Prove the following statements: 1. $A \subset B \implies \mathring{A} \subset \mathring{B}$ 2. $A \subset B \implies \overline{A} \subset \overline{B}$*

Question 4. *Let $S_1, S_2, S_3, S_4 \subset \mathbb{R}^2$ be defined by*

$$S_1 = [1,3] \times [0,4] \qquad S_2 = \left\{(x,y) \in \mathbb{R}^2 \; : \; (x+2)^2 + (y-2)^2 < 4\right\}$$

$$S_3 = \left\{(x,y) \in \mathbb{R}^2 \; : \; x^2 + y^2 = 4\right\} \qquad S_4 = \left\{(t^3, -t^2) \; : \; t \in \mathbb{R}\right\}$$

1. *Draw S_1, S_2, S_3, S_4.*
2. *Prove that S_1, S_2, S_3, S_4 are path-connected.*
3. *Prove that $S_1 \cup S_2 \cup S_3 \cup S_4$ is path-connected.*

Question 5. *Prove that $S = \left\{(x,y) \in \mathbb{R}^2 \; : \; 1 < x^2 + y^2 \leq 4\right\}$ is path-connected.*

Question 6. *Let $S = \left\{\mathbf{x} \in \mathbb{R}^n \; : \; \|\mathbf{x}\| = 1\right\}$ and $f : S \to \mathbb{R}$ be a continuous function.*

1. *(★) Prove that S is path-connected.*
2. *(★) Prove that there exists $\mathbf{x} \in S$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.*
3. *Prove that S is compact.*
4. *Prove that f has a min and a max.*

Question 7. *Are the sets from Question 1 and Question 2 open? closed? bounded? compact?*

Question 8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function.*

1. *Guess and write the formal definition (i.e. $\varepsilon - \delta$-style) of $\lim_{\|\mathbf{x}\| \to +\infty} f(\mathbf{x}) = +\infty$.*
2. *(★) Prove that if $\lim_{\|\mathbf{x}\| \to +\infty} f(\mathbf{x}) = +\infty$ then f has a min.*

Question 9. *Let $(S_k)_{k \in \mathbb{N}_{\geq 0}}$ be a decreasing sequence of non-empty compact subsets of $\mathbb{R}^n$: $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots$*

1. *(★) Prove that $\bigcap_{k \in \mathbb{N}_{\geq 0}} S_k \neq \varnothing$*
2. *Can the above intersection be empty if we replace compact by closed?*
3. *Can the above intersection be empty if we replace compact by bounded?*

Question 10 (Arithmetico–geometric sequences).
Let $(u_n)_{n\in\mathbb{N}_{\geq 0}}$ be a real-valued sequence satisfying

$$\forall n \in \mathbb{N}_{\geq 0},\ u_{n+1} = \alpha u_n + \beta$$

1. *We assume that $\alpha = 1$. Find an algebraic expression for u_n (in terms of u_0).*
2. *We assume that $\alpha \neq 1$.*
   (a) *Find $r \in \mathbb{R}$ such that $v_n = u_n - r$ is a geometric sequence.*
   (b) *Deduce an algebraic expression for u_n (in terms of u_0).*

Question 11. *We define a sequence $(\mathbf{u}_n)_{n\geq 0}$ in $\mathbb{R}^2$ by $\mathbf{u_0} = (1,0)$ and $\mathbf{u}_{n+1} = \frac{1}{2}\mathbf{u}_n + (1,2)$.*
Is it convergent? If so, what is its limit?

Question 12. *Let $S = \left\{(x,y) \in \mathbb{R}^2 \ :\ x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q}\right\}$.*

1. *Without computing its interior or closure, tell whether S is closed and/or open.*
2. *Compute $\mathring{S}$, $\overline{S}$ and ∂S.*
   We recall that (see the notes from Oct. 1)
   - $\forall x_1, x_2 \in \mathbb{R},\ x_1 < x_2 \implies \left(\exists q \in \mathbb{Q},\ x_1 < q < x_2\right)$
   - $\forall x_1, x_2 \in \mathbb{R},\ x_1 < x_2 \implies \left(\exists r \in \mathbb{R}\setminus\mathbb{Q},\ x_1 < r < x_2\right)$

Question 13. *Let $f : (a,b) \to \mathbb{R}$ be a function defined on an open interval and let $x_0 \in (a,b)$.*

1. *Recall the definition of "f is differentiable at x_0" from MAT135/137/157.*
2. *Recall the definition of "f is differentiable at x_0" from MAT237.*
3. *Prove that f is differentiable at x_0 in the sense of MAT1XX if and only if f is differentiable at x_0 in the sense of MAT237.*
4. *If f is differentiable at x_0, how are the derivative of f at x_0 and the differential of f at x_0 related?*

Question 14. *Are the following functions differentiable? If so determine their differentials and gradients.*

1. $f(x,y) = e^{xy}(x+y)$ *on* $\mathbb{R}^2$.
2. $f(x,y,z) = xy + yz + zx$ *on* $\mathbb{R}^3$.

Question 15. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by* $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

1. *Is f continuous?*
2. *Do the partial derivatives of f exist? Are they continuous?*
3. *Is f differentiable?*

Question 16. *Same questions as above for $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by* $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

Question 17. *Prove that the following functions $\mathbb{R}^2 \to \mathbb{R}$ admit directional derivatives at $(0,0)$ along any direction but are not continuous at $(0,0)$.*

1. $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$
2. $g(x,y) = \begin{cases} y^2\ln(|x|) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Question 18. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sqrt{x^2+y^2}$.*

1. *Prove that f is differentiable at $(4,3)$ and determine its differential at $(4,3)$.*
2. *Give an approximation of $\sqrt{4.05^2 + 2.93^2}$.*

($\bigstar$) $\equiv$ advanced question.