Sequences in $\mathbb{R}^m$

Def: A sequence in $\mathbb{R}^m$ is a function $\{k \in \mathbb{N} : k \geq k_0\} \longrightarrow \mathbb{R}^m$, $k \longmapsto a_k$

We use the notation $(a_k)_{k \geq k_0}$

The online notes use $\{a_k\}_{k \geq k_0}$, but I prefer (a_k) since the order matters

Def: We say that a sequence $(a_k)_{k \geq k_0}$ in $\mathbb{R}^m$ converges to $L \in \mathbb{R}^m$ if

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, \quad k \geq K \Rightarrow \|a_k - L\| < \varepsilon$$

denoted by $\lim\limits_{k \to +\infty} a_k = L$

Remark: $\lim\limits_{k \to +\infty} a_k = L \Leftrightarrow \lim\limits_{k \to +\infty} \overbrace{\|a_k - L\|}^{\text{real valued sequence}} = 0$

The proof is the same as the one for functions

Theorem: Let $(a_k)_{k \geq k_0}$ be a sequence in $\mathbb{R}^m$. Denote $a_k = (a_{k1}, \ldots, a_{km})$.

Then $\lim\limits_{k \to +\infty} a_k = L \Leftrightarrow \forall i = 1, \ldots, m, \ \lim\limits_{k \to +\infty} a_{ki} = L_i$

Again, it's enough to understand well the real valued case to compute limits

Ex: $\left(\frac{1}{k}, \frac{2k^2}{k^2+1}\right) \xrightarrow[k \to +\infty]{} (0, 2)$ since $\begin{cases} \lim\limits_{k \to +\infty} \frac{1}{k} = 0 \\ \lim\limits_{k \to +\infty} \frac{2k^2}{k^2+1} = 2 \end{cases}$

Theorem: Let $(a_k)_{k \geq k_0}$ be a convergent sequence in $\mathbb{R}^m$ and $S \subset \mathbb{R}^m$

If $\forall k \geq k_0$, $a_k \in S$ then $\lim\limits_{k \to +\infty} a_k \in \overline{S}$

Δ Denote $L = \lim\limits_{k \to +\infty} a_k$, let $\varepsilon > 0$, then $\exists K \in \mathbb{N}$ s.t. $\|a_K - L\| < \varepsilon$, ie $a_K \in B(L, \varepsilon)$.

Hence $B(L, \varepsilon) \cap S \neq \emptyset$. Furthermore $L \in \overline{S}$ $\square$

Ex: A sequence "can't escape" from a closed set.

Ex: $\{x \in \mathbb{R}^2 : \|x\| \notin \mathbb{Q}\}$ is not closed: $a_m = \left(\frac{1}{m}, \frac{1}{m}\right)$

Ex: $\{(x,y) : |x| < 1, |y| < 1\}$ not closed $\left(1 - \frac{1}{n}, 1\right)$

Def: We say that a sequence $(a_k)_{k \geq k_0}$ is bounded if
$$\exists M > 0, \forall k \in \mathbb{N}_{\geq k_0}, \|a_k\| < M$$

Def: A subsection of a sequence $(a_k)_{k \geq k_0}$ is a sequence $(a_{\varphi(j)})_{j \in \mathbb{N}}$ where $\varphi : \mathbb{N} \to \{k \in \mathbb{N} : k \geq k_0\}$ is increasing.

⚠ in the notes they write a_{k_j} for $a_{\varphi(j)}$, but it may be confusing.

Intuitively, we omit some terms

$$\cancel{a_{k_0}} \quad \underset{a_{\varphi(0)}}{\overset{a_{k_0+1}}{\|}} \quad \cancel{a_{k_0+2}} \quad \cancel{a_{k_0+3}} \quad \underset{a_{\varphi(1)}}{\overset{a_{k_0+4}}{\|}} \quad \underset{a_{\varphi(2)}}{\overset{a_{k_0+5}}{\|}} \quad \cancel{a_{k_0+6}} \quad \underset{a_{\varphi(3)}}{\overset{a_{k_0+7}}{\|}} \quad \cancel{a_{k_0+8}} \quad \cancel{a_{k_0+9}} \ldots$$

$\varphi(0) = k_0+1, \ \varphi(1) = k_0+4, \ldots$

Ex.: Let $a_n = (-1)^n$ for $n \in \mathbb{N}$ be a sequence in $\mathbb{R}$, then $a_{2n+1} = -1$ is a subsequence of (a_n)

$\cancel{1}\ -1\ \cancel{1}\ -1\ \cancel{1}\ -1\ \ldots$

→ the contrapositive may be useful.

Lemma: if $\lim\limits_{k \to +\infty} a_k = L$ then $\lim\limits_{j \to +\infty} a_{\varphi(j)} = L$ (for any subsequence)

"Any subsequence of a convergent sequence converges to the same limit".

Theorem: A bounded sequence $(a_k)_{k \geq k_0}$ in $\mathbb{R}^n$ admits a convergent subsequence.

Δ • the first component a_{k1} of a_k is bounded so by the real case $\exists \varphi_1$ s.t. $a_{\varphi_1(j),1}$ is convergent

• we repeat the process to the second component of $(a_{\varphi_1(j)})$, then we get φ_2 s.t. the first two components of $(a_{\varphi_1(\varphi_2(j))})$ are CV

• And so on

$\square$

Ex.: $a_n = (-1)^n$ is bounded, not CV, but $a_{2n} = 1$ CV

Compactness

Def: A subset $S \subset \mathbb{R}^m$ is compact if any sequence with elements in S admits a subsequence which is convergent in S
↳ the limit of the subsequence is in S

⚠ That's not the usual definition, but it's equivalent for $\mathbb{R}^m$

Theorem (Bolzano-Weierstrass) S compact $\Leftrightarrow$ S closed + bounded

Δ $\Leftarrow$ Assume that S is closed and bounded and let (a_k) be a sequence with values in S.
Then (a_k) is bounded and admits a CV subsequence $(a_{\varphi(j)})$ with limit L. ↳ by the last theorem
$L \in \overline{S} = S$ → since S is closed
↳ by a previous theorem

$\Rightarrow$ by contrapositive

1st case: S is not bounded
$\forall k \in \mathbb{N}, \exists a_k \in S, \|a_k\| > k$
then any subsequence of (a_k) satisfies $\|a_{\varphi(j)}\| \xrightarrow[j\to+\infty]{} +\infty$

2nd case: S is not closed, ie $S \subsetneq \overline{S}$
$\exists L \in \overline{S} \setminus S$
$\forall k \in \mathbb{N}, \exists a_k \in B(L, 1/k) \cap S$
then $\|a_k - L\| < \frac{1}{k} \xrightarrow[k\to\infty]{} 0$ so (a_k) converges to $L \notin S$
and any subsequence converges to $L \notin S$ □

Theorem: The continuous image of a compact set is compact.
ie: if $f: \mathbb{R}^m \to \mathbb{R}^k$ is continuous and $S \subset \mathbb{R}^m$ compact then $f(S) \subset \mathbb{R}^k$ compact

Δ Let (a_k) be a sequence in $f(S)$, then $a_k = f(b_k)$ for $b_k \in S$

Next, (b_k) admits a CV subsequence $(b_{\varphi(j)})$ with limit $L \in S$

Homework: $\lim_{j\to+\infty} a_{\varphi(j)} = f(L) \in f(S)$ □

Homework: questions in section 1.3 of the online notes

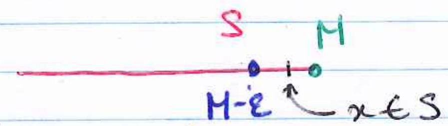
→ A compact set of $\mathbb{R}$ admits a supremum / infimum (which is in S)!
nonempty

Proposition: If $S \subset \mathbb{R}$, $S \neq \emptyset$ is compact then $\sup S \in S$ and $\inf S \in S$

(Δ n=1)

Δ ① Since S is bounded, $M = \sup S$ exists by the LUB principle

② Let $\varepsilon > 0$. Then $M - \varepsilon$ is not an upper bound of S so there exists $x \in S$ s.t. $M - \varepsilon < x$



so $S \cap (M-\varepsilon, M+\varepsilon) \neq \emptyset$

Hence $\sup S = M \in \bar{S} = S$ since S is closed $\square$

Corollary (EVT) Let $K \subset \mathbb{R}^m$ be a compact set and $f: K \longrightarrow \mathbb{R}$ be a continuous function.

Then f has a min and a max
ie: $\exists c, d \in K, \forall x \in K, \; f(c) \leq f(x) \leq f(d)$

Δ By a previous theorem $f(K)$ is compact.

Since $f(K) \subset \mathbb{R}$, by the previous proposition $\exists m, M \in f(K)$ such that $\forall x \in K$, $m \leq f(x) \leq M$.

Since $m \in f(K)$, $\exists c \in K$ st $m = f(c)$, similarly for M $\square$

The EVT from MAT137 is a particular case of the above since $[a,b]$ a segment line is compact.

Homework: questions from 1.4 of the online notes