Dedekind-completeness of $\mathbb{R}$: A detour via MAT137

UNIVERSITY OF TORONTO

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Dedekind-completeness of $\mathbb{R}$

The following results seen in MAT137 about $\mathbb{R}$ are equivalent:

- The Least Upper Bound principle
- The Monotone Convergence Theorem for sequences
- The Extreme Value Theorem
- The Intermediate Value Theorem
- Rolle's Theorem/The Mean Value Theorem
- A **bounded** sequence in $\mathbb{R}$ admits a convergent subsequence
- Cuts (if you took MAT157):
$$\left.\begin{array}{r} A, B \neq \emptyset \\ \mathbb{R} = A \cup B \\ \forall a \in A, \forall b \in B, a < b \end{array}\right\} \implies \exists! c \in \mathbb{R}, \forall a \in \mathbb{R}, \forall b \in B, a \leq c \leq b$$
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(You can safely ignore that:)
$\mathbb{R}$ *is the unique Dedekind-complete (totally) ordered field.*

How to understand Dedekind-Completeness

Intuitively, the Dedekind-completeness of the real line tells us two things about it:

1. There is no infinitely small positive real number (*Archimedean property*): $\forall \varepsilon > 0,\ \forall A > 0,\ \exists n \in \mathbb{N}_{>0},\ n\varepsilon > A$.

   ▲ Assume, for the sake of contradiction, that

   $$\exists \varepsilon > 0,\ \exists A > 0,\ \forall n \in \mathbb{N}_{>0},\ n\varepsilon \leq A$$

   Then $S = \{n\varepsilon \ :\ n \in \mathbb{N}_{>0}\}$ is bounded from above by A.
   So it admits a least upper bound $M = \sup S$.
   Since $M - \varepsilon$ is less than the least upper bound M of S, it is not an upper bound of S, i.e. $M - \varepsilon < n\varepsilon$ for some $n \in \mathbb{N}_{>0}$.
   But then $M < (n+1)\varepsilon$, which is not possible since $(n+1)\varepsilon \in S$ and M is an upper bound of S.
   Contradiction. ■

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1. There is no infinitely small positive real number (*Archimedean property*): $\forall \varepsilon > 0, \forall A > 0, \exists n \in \mathbb{N}_{>0}, n\varepsilon > A$.

2. There is no gap in the real line.
   For instance,
   - LUB: $\sqrt{2} = \sup \left\{ x \in \mathbb{Q} \ : \ x^2 < 2 \right\}$.
   - MCT: define a sequence by $x_0 = 1$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$.
     Then (x_n) converges to some limit l by the MCT.
     But this limit must satisfy $l^2 = 2$.
   - IVT: let $f(x) = x^2 - 2$. Then $f(0) < 0$ and $f(2) > 0$.
     Hence we deduce from the IVT that f has a root,
     i.e. $\exists x \in \mathbb{R}, x^2 - 2 = 0$.

Consequences

The Dedekind-completeness of the real line has several consequences that you already know:

- The various results connecting the sign of f' to the monotonicity of f.
- $ACV \implies CV$ (for series and improper integrals).
- The Fundamental Theorem of Calculus.
- L'Hôpital's rule.
- The BCT and the LCT (for series and improper integrals).
- Cauchy-completeness of $\mathbb{R}$: *any Cauchy sequence converges*.
- ...

In some sense, MAT137 was about the Dedekind-completeness of the real line and its consequences.

What about $\mathbb{R}^n$?

The statements about the Dedekind-completeness of the real line can **not** be extended to $\mathbb{R}^n$ simply by replacing $\mathbb{R}$ with $\mathbb{R}^n$ since they use the following properties of $\mathbb{R}$:

- $\mathbb{R}$ has a product and any non-zero real number has an inverse.
- $\mathbb{R}$ has an order *compatible* with its addition and product.

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Any bounded sequence in $\mathbb{R}^n$ admits a convergent subsequence.

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Warning: in the online lecture notes of the course, the above result is called *Completeness of $\mathbb{R}^n$*.
Be careful that when people say that "*some space is complete*", they usually talk about a strictly weaker result (Cauchy-completeness).

Appendix: LUB $\implies$ BW (you can safely ignore it)

How can we deduce from the LUB principle that a *bounded real valued sequence* $(a_k)_k$ *admits a convergent subsequence* $\left(a_{\varphi(j)}\right)_j$?

▲ We know that $\exists M > 0, \forall k, |a_k| < M$.
Then $L = \sup\left\{x \in [-M, M] \ : \ x < a_k \text{ for infinitely many } k\right\}$ exists by the LUB principle (the set is bounded from above by M and contains $-M$).

Assume that $\varphi(j-1)$ is constructed and we want to construct $\varphi(j)$.
Since $L - \frac{1}{j}$ is less than L, it is not an upper bound of the above set.
Hence there exist infinitely many k such that $L - \frac{1}{j} < a_k < L + \frac{1}{j}$ *(check it)*.
We pick $\varphi(j)$ to be such a k which is greater than $\varphi(j-1)$ (which is possible since there are infinitely many such k).
Hence we constructed a subsequence $\left(a_{\varphi(j)}\right)_j$ such that

$$\forall j,\ L - \frac{1}{j} < a_{\varphi(j)} < L + \frac{1}{j}$$

Therefore this subsequence tends to L. ■