University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

In class questions

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Question 1. *Study* $\lim_{(x,y)\to(0,0)} f(x,y)$ *for the following functions.*

1. $f(x,y)=\frac{x^3y}{x^4+y^2}$ *defined on* $\mathbb{R}^2\setminus\{(0,0)\}$.

   $\left|\frac{x^3y}{x^4+y^2}\right| = \frac{|x^2y|}{x^4+y^2}|x| \le \frac{1}{2}|x| \xrightarrow[(x,y)\to(0,0)]{} 0$

   Hence $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0.$

2. $f(x,y)=\frac{xy^4}{x^2+y^8}$ *defined on* $\mathbb{R}^2\setminus\{(0,0)\}$.

   $f(y^4,y)=\frac{1}{2} \xrightarrow[y\to 0]{} \frac{1}{2}$
   $f(0,y)=0 \xrightarrow[y\to 0]{} 0$

   Hence $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ *DNE.*

3. $f(x,y)=\frac{1-\cos(\sqrt{|xy|})}{|y|}$ *defined on* $\{(x,y)\in\mathbb{R}^2 \;:\; y\neq 0\}$.
   Hint: *study* $g(x)=1-\cos(x)-x^2$.

   One can check that $\forall x\in\mathbb{R},\ 1-\cos(x)\le x^2$.
   Then $0 \le \frac{1-\cos\left(\sqrt{|xy|}\right)}{|y|} \le \frac{|xy|}{|y|} = |x| \xrightarrow[(x,y)\to(0,0)]{} 0.$

   Hence $\lim_{(x,y)\to(0,0)} \frac{1-\cos(\sqrt{|xy|})}{|y|} = 0.$

4. $f(x,y)=\frac{\frac{y^3-x^8y}{x^6+y^2}-y}{\sqrt{x^2+y^2}}$ *defined on* $\mathbb{R}^2\setminus\{(0,0)\}$.

   $$\left|\frac{\frac{y^3-x^8y}{x^6+y^2}-y}{\sqrt{x^2+y^2}}\right| = \frac{|x^3y|}{x^6+y^2}\cdot\frac{|x|}{\sqrt{x^2+y^2}}\cdot x^2\cdot(1+x^2).$$

   The first factor is bounded from above by $\frac{1}{2}$, *the second one is bounded from above by* 1, *the third one converges to* 0 *and the last one converges to* 1 *when* $(x,y)\to(0,0)$.

Hence $\lim_{(x,y)\to(0,0)} \frac{\frac{y^3-x^8y}{x^6+y^2}-y}{\sqrt{x^2+y^2}} = 0.$

5. $f(x,y) = \frac{3x^2+xy}{\sqrt{x^2+y^2}}$ *defined on* $\mathbb{R}^2 \setminus \{(0,0)\}$.

   Notice that $x^2 \leq x^2+y^2 = \|(x,y)\|^2$ *and that* $|xy| \leq \frac{x^2+y^2}{2} = \frac{\|(x,y)\|^2}{2}$.

   Hence $\left|\frac{3x^2+xy}{\sqrt{x^2+y^2}}\right| \leq \frac{3x^2+|xy|}{\sqrt{x^2+y^2}} \leq \frac{3\|(x,y)\|^2+\frac{\|(x,y)\|^2}{2}}{\|(x,y)\|} = 3\|(x,y)\| + \frac{\|(x,y)\|}{2} \xrightarrow[(x,y)\to(0,0)]{} 0.$

   Furthermore $\lim_{(x,y)\to(0,0)} \frac{3x^2+xy}{\sqrt{x^2+y^2}} = 0.$

6. $f(x,y) = \frac{(x^2-y)(y^2-x)}{x+y}$ *defined on* $\{(x,y)\in\mathbb{R}^2 \,:\, x+y \neq 0\}$.

   $f(x,0) = -x^2 \xrightarrow[x\to 0]{} 0.$

   $f(x,-x+x^2) = x^3-2x^2+x-1 \xrightarrow[x\to 0]{} -1.$

   Hence $\lim_{(x,y)\to(0,0)} \frac{(x^2-y)(y^2-x)}{x+y}$ *DNE.*

Question 2. *Are the following subsets open? closed?*

1. $S = \left\{(x,y)\in\mathbb{R}^2 \,:\, x^2y^2 > 1\right\} \subset \mathbb{R}^2$

   We may write $S = f^{-1}((1,+\infty))$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *is defined by* $f(x,y) = x^2y^2$.
   Notice that f *is continuous and that* $(1,+\infty) \subset \mathbb{R}$ *is open.*
   Hence S *is open as the inverse image of an open set by a continuous map.*

2. $T = \left\{(x,y)\in\mathbb{R}^2 \,:\, x^2+y^2 = 1,\ y \geq 0\right\} \subset \mathbb{R}^2$

   Notice that $T = f^{-1}(\{1\}) \cap g^{-1}([0,+\infty))$ *where* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *are respectively defined by* $f(x,y) = x^2+y^2$ *and* $g(x,y) = y$.
   Each of the both preimages is closed as the inverse image of a closed subset by a continuous map.
   Hence T *is closed as the intersection of closed sets.*