University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

Solutions to the in class questions

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Question: Let $S \subset \mathbb{R}^n$. Prove that $\mathring{S}$ is open.

Answer.
Let $x \in \mathring{S}$ then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset S$.
I claim that $B(x, \varepsilon) \subset \mathring{S}$.
Indeed, let $y \in B(x, \varepsilon)$ then we can check (triangle inequality) that $B(y, \varepsilon - \|x - y\|) \subset B(x, \varepsilon) \subset S$.
So for any $y \in B(x, \varepsilon)$, there exists $\varepsilon' = \varepsilon - \|x - y\| > 0$ such that $B(y, \varepsilon') \subset S$ so $y \in \mathring{S}$.
Hence $B(x, \varepsilon) \subset \mathring{S}$ as claimed.

To sum up, for any $x \in \mathring{S}$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \mathring{S}$.
Hence $\mathring{S}$ is open. ∎

Remark: *we also proved that an open ball is open.*

Question: Let $S \subset \mathbb{R}^n$. Prove that $\overline{S}$ is closed.

Answer. We are going to prove that $\left(\overline{S}\right)^c = \mathbb{R}^n \setminus S$ is open.

$$\begin{aligned}
\left(\overline{S}\right)^c &= \left\{x \in \mathbb{R}^n \ :\ \text{no}\,(\forall \varepsilon > 0,\ B(x, \varepsilon) \cap S \neq \varnothing)\right\} \\
&= \left\{x \in \mathbb{R}^n \ :\ \exists \varepsilon > 0,\ B(x, \varepsilon) \cap S = \varnothing\right\} \\
&= \left\{x \in \mathbb{R}^n \ :\ \exists \varepsilon > 0,\ B(x, \varepsilon) \subset S^c\right\} \\
&= \overset{\circ}{\widehat{S^c}}
\end{aligned}$$

This last set is open according to the first question. ∎

Remarks: *We also proved that the complement of the closure is the interior of the complement:*

$$\left(\overline{S}\right)^c = \overset{\circ}{\widehat{S^c}}$$

By replacing S by S^c we get that the complement of the interior is the closure of the complement:

$$\left(\mathring{S}\right)^c = \overline{S^c}$$

These two properties can be very helpful!

Advanced question:

1. Let $(O_i)_{i\in I}$ be a family of open subsets of $\mathbb{R}^n$. Prove that $\bigcup_{i\in I} O_i$ is open.

   Answer.
   Let $x \in \bigcup_{i\in I} O_i := \left\{x \in \mathbb{R}^n \;:\; \exists i \in I,\, x \in O_i\right\}$.
   Then there exists $i \in I$ such that $x \in O_i$.
   Since O_i is open there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset O_i$.
   Hence $B(x,\varepsilon) \subset O_i \subset \bigcup O_i$.
   We've just proved that for any $x \in \bigcup O_i$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset \bigcup O_i$.
   Furthermore $\bigcup O_i$ is open. ■

   Remark: *notice that the above proof is true for any I (finite or not, countable or not).*

2. Let $U, V \subset \mathbb{R}^n$ be two open sets. Prove that $U \cap V$ is open.

   Answer.
   Let $x \in U \cap V$.
   Since $x \in U$ open, there exists $\varepsilon_1 > 0$ such that $B(x,\varepsilon_1) \subset U$.
   Since $x \in V$ open, there exists $\varepsilon_2 > 0$ such that $B(x,\varepsilon_2) \subset V$.
   Take $\varepsilon = \min(\varepsilon_1,\varepsilon_2) > 0$.
   Then $B(x,\varepsilon) \subset B(x,\varepsilon_1) \subset U$ and $B(x,\varepsilon) \subset B(x,\varepsilon_2) \subset V$.
   Hence $B(x,\varepsilon) \subset U \cap V$.
   We've just proved that for any $x \in U \cap V$ there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U \cap V$.
   Furthermore $U \cap V$ is open. ■

3. Find a infinite family of open subsets of $\mathbb{R}$ whose intersection is not open.

   Answer.
   $$\bigcap_{n\in\mathbb{N}_{>0}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$
   ■

   Important remark: *according to question 2, a finite intersection of open sets is open. But, according to question 3, it is possible for an infinite intersection of open sets to* ***not*** *be open.*

4. What about closed sets?

   Answer.
   Using the fact that S is closed if and only if S^c is open, we get that:

   - Any intersection (finite or not) of closed sets is closed.
   - A **finite** union of closed sets if closed.
   - However, an infinite union of closed sets may not be closed:
   $$\bigcup_{n\in\mathbb{N}>0} \left[\frac{1}{n}, +\infty\right) = (0, +\infty)$$
   or
   $$\bigcup_{x\in(-1,1)} \{x\} = (-1, 1)$$

   ■