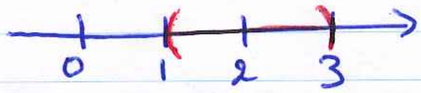
SOME TOPOLOGICAL NOTIONS

Terminology: Balls and Spheres

Def: For $a \in \mathbb{R}^m$ and $r \in \mathbb{R}_{>0}$, the open ball centered at a with radius r is:

$$B(a,r) := \{x \in \mathbb{R}^m : \|x-a\| < r\} \subset \mathbb{R}^m$$

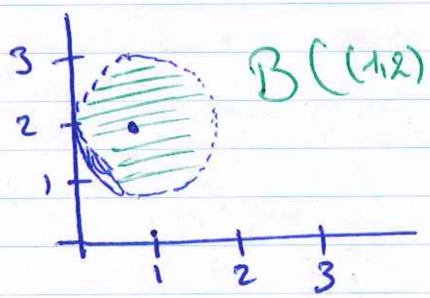
Ex:



$B(2,1) = (1,3) \subset \mathbb{R}$

$\overline{B}(2,1) = [1,3] \subset \mathbb{R}$

$S(2,1) = \{1,3\} \subset \mathbb{R}$



$B((1,2),1) \subset \mathbb{R}^2$

Def: the closed ball centered at a with radius r is

$$\overline{B}(a,r) := \{x \in \mathbb{R}^m : \|x-a\| \leq r\} \subset \mathbb{R}^m$$

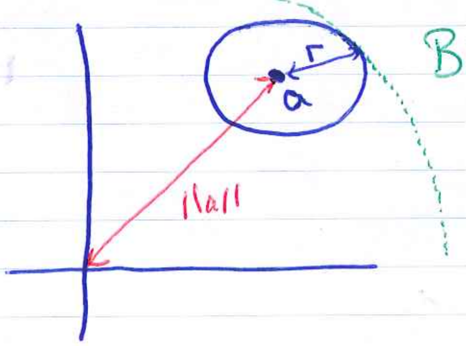
Def: the sphere centered at a with radius r is

$$S(a,r) := \{x \in \mathbb{R}^m : \|x-a\| = r\}$$

Def: A subset $S \subset \mathbb{R}^m$ is bounded if there exists $r \in \mathbb{R}_{>0}$ st. $S \subset B(0,r)$

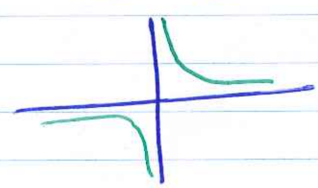
ie $\exists r > 0, \forall x \in S, \|x\| < r$

Ex: $B(a,r) \subset \mathbb{R}^m$ is bounded



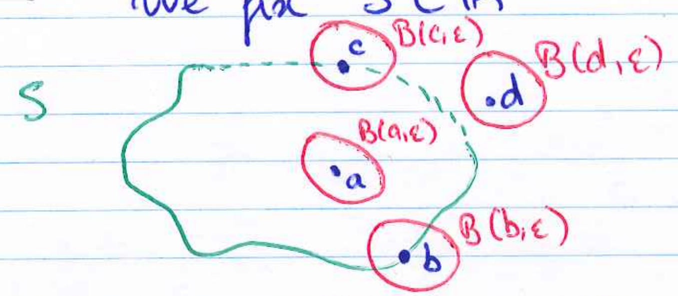
$B(a,r) \subset B(0, \|a\|+r)$

Ex: $\{(x,y) : xy = 1\}$ is not bounded



Terminology: Interior, closure, boundary

We fix $S \subset \mathbb{R}^m$



$a \in S$	$a \in \mathring{S}$	$a \in \overline{S}$	$a \notin \partial S$
$b \in S$	$b \notin \mathring{S}$	$b \in \overline{S}$	$b \in \partial S$
$c \notin S$	$c \notin \mathring{S}$	$c \in \overline{S}$	$c \in \partial S$
$d \notin S$	$d \notin \mathring{S}$	$d \notin \overline{S}$	$d \notin \partial S$

Def: We say that $x \in \mathbb{R}^m$ is an *interior point* of S if there exists $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset S$

Notation: the interior of S is $\mathring{S} := \{x \in \mathbb{R}^m : \exists \varepsilon > 0,\ B(x,\varepsilon) \subset S\}$
or S^{int}

Def: we say that $x \in \mathbb{R}^m$ is a *closure point* (or *adherent point*) of S if $\forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset$

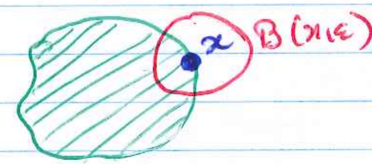
Notation: the closure of S is $\overline{S} := \{x \in \mathbb{R}^m : \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset\}$

Theorem: $\mathring{S} \subset S \subset \overline{S}$

Δ $\mathring{S} \subset S$: let $x \in \mathring{S}$ then $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset S$
hence $x \in B(x,\varepsilon) \subset S$
so $x \in \mathring{S} \Rightarrow x \in S$

$S \subset \overline{S}$: let $x \in S$.
For any $\varepsilon > 0$, $x \in S \cap B(x,\varepsilon) \neq \emptyset$, so $x \in \overline{S}$ $\square$

Def: the *boundary* of S is $\partial S := \overline{S} \setminus \mathring{S}$



Prop: $\overline{S} = S \cup \partial S$ and $\partial S \cap \mathring{S} = \emptyset$

Theorem: $x \in \partial S \Leftrightarrow \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset$ and $B(x,\varepsilon) \cap S^c \neq \emptyset$

Δ $x \in \overline{S} \Leftrightarrow \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset$
$x \notin \mathring{S} \Leftrightarrow$ no $(\exists \varepsilon > 0,\ B(x,\varepsilon) \subset S) \Leftrightarrow \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S^c \neq \emptyset$ $\square$

Cor: $\partial S = \partial(S^c)$

Δ Notice that $(S^c)^c = S$ $\square$