PRELIMINARIES

Cartesian product

<u>Def</u>: An n-tuple is an ordered list of n elements $(x_1, \dots, x_n)$

<u>Rem</u>: couple ≡ 2-tuple triple ≡ 3-tuple

<u>Fundamental property</u>: $(x_1, \dots, x_n) = (y_1, \dots, y_n) \iff \forall i,\ x_i = y_i$

<u>Rem</u>: ① $\{1,2,3\} = \{3,2,1\}$ (Sets)
but $(1,2,3) \neq (3,2,1)$ (Tuple)

② $\{1,2,2,3\} = \{1,2,3\}$
but $(1,2,2,3) \neq (1,2,3)$

<u>Def</u>. Given 2 sets A and B: $A \times B = \{(a,b) : a \in A,\ b \in B\}$

<u>Ex</u>: $A = \{\pi, e\}$, $B = \{1, \sqrt{2}, \pi\}$

$$A \times B = \{(\pi,1), (\pi,\sqrt{2}), (\pi,\pi), (e,1), (e,\sqrt{2}), (e,\pi)\}$$

<u>Rem</u>: if A and B are finite then $\#(A \times B) = \#A \cdot \#B$

<u>Def</u>: $A_1 \times A_2 \times \dots \times A_n = \{(a_1, \dots, a_n) : a_i \in A_i\}$

<u>Rem</u>: We will often identify the following sets:

$$\begin{array}{ccc} (A \times B) \times C & A \times (B \times C) & A \times B \times C \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ ((a,b),c) & (a,(b,c)) & (a,b,c) \end{array}$$

even if they are not formally the same set.

Functions

informal definition

Def: A function (or map, or mapping) is the data of two sets A and B together with a "process" that associates to each element $x \in A$ a unique element $f(x) \in B$

Notation: $f : A \longrightarrow B$

name, domain, codomain

Notation: let $f : A \to B$ be a function

① the image of $E \subset A$ by f is $f(E) := \{ f(x) : x \in E \}$

② the preimage of $F \subset B$ by f is $f^{-1}(F) := \{ x \in A : f(x) \in F \}$

Def: the graph of $f : A \longrightarrow B$ is $\Gamma_f := \{ (x,y) \in A \times B : y = f(x) \}$

Rem: a function is entirely determined by its graph

Def: $f : A \longrightarrow B$ is injective (or 1-to-1) if $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or equivalently (contrapositive) $\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Def: $f : A \longrightarrow B$ is surjective (or onto) if $\forall y \in B, \exists x \in A,\ y = f(x)$

Def: $f : A \to B$ is bijective if it is injective and surjective

ie $\forall y \in B, \exists! x \in A,\ y = f(x)$

Prop: $f : A \longrightarrow B$ is bijective iff $\exists g : B \longrightarrow A$ such that

$$\begin{cases} g \circ f = id_A \\ f \circ g = id_B \end{cases}$$

Then we say that g is the inverse of f, denoted f^{-1}

Ex: Slides

Geometry of $\mathbb{R}^n$

Def: $\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \{ (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$

Rem: ① the x_i are bound variables, however, we will often use:

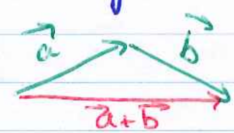
(x,y) for $n=2$
(x,y,z) for $n=3$
$(x_1, \dots, x_n)$ for $n>3$

② In the online notes, an element of $\mathbb{R}^n$ is written in bold, you can also use an arrow to avoid any confusion

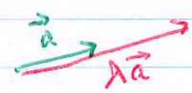
$$\vec{v} = (x_1, \dots, x_n)$$

For $a=(a_1, \dots, a_n)$, $b=(b_1, \dots, b_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define

Addition: $a+b := (a_1+b_1, a_2+b_2, \dots, a_n+b_n) \in \mathbb{R}^n$



Scalar multiplication: $\lambda a := (\lambda a_1, \dots, \lambda a_n) \in \mathbb{R}^n$



Notation: ① $\vec{e_1} = (1,0,\dots,0)$, $\vec{e_2} = (0,1,0,\dots,0)$, $\dots$, $\vec{e_n} = (0,\dots,0,1)$ in $\mathbb{R}^n$

② $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$ in $\mathbb{R}^3$

Def: (dot product) $\underset{\mathbb{R}^n}{a} \cdot \underset{\mathbb{R}^n}{b} := a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \in \mathbb{R}$: it takes 2 vectors and gives 1 scalar

Prop: for $a, b, c \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

① $a \cdot b = b \cdot a$ (commutativity)
② $(\lambda a + b) \cdot c = \lambda (a \cdot c) + b \cdot c$ (bilinearity)
③ $a \neq 0 \Rightarrow a \cdot a > 0$ (positive definite)

} the dot product is an inner-product

④ $a \cdot a = 0 \Rightarrow a = 0$
⑤ $0 \cdot a = 0$

Ex: $(1,2)\cdot(-1,3)=5$

$(1,0,3)\cdot(-1,1,-1)=-4$

$(1,-1,1,-1)\cdot(1,0,2,-1)=4$

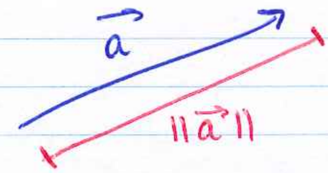
but $(1,1,1)\cdot(1,2)$ is not defined

or $|a|$ in the online notes

Def: (Euclidean norm)
For $a\in\mathbb{R}^n$, we denote $\|a\|:=\sqrt{a\cdot a}=\sqrt{a_1^2+\dots+a_n^2}$

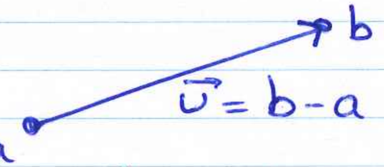
Geometric interpretation

① $\|\vec{a}\|$ is the length of $\vec{a}$ (or magnitude)



② $\|b-a\|$ is the distance between a and b

$\| \; = \sqrt{(b_1-a_1)^2+\dots+(b_n-a_n)^2}$



Rem: An element of $\mathbb{R}^n$ may represent a vector (velocity, force) or a point (position)

Prop: for $a,b,c\in\mathbb{R}^n$, $\lambda\in\mathbb{R}$

① $\|a\|\geqslant 0$

② $\|a\|=0\Rightarrow a=0$ (positive definite)

③ $\|\lambda a\|=|\lambda|\cdot\|a\|$ (positive homogeneity)

④ $\|a+b\|\leqslant\|a\|+\|b\|$ (triangle inequality)

$\|\cdot\|$ is a norm

2, 3, 4 $\Rightarrow$ 1

⑤ $|a\cdot b|\leqslant\|a\|\cdot\|b\|$ (Cauchy-Schwarz inequality)

⑥ $a\cdot e_j=a_j$, $e_j\cdot e_j=1$, $e_i\cdot e_j=0$ for $i\neq j$

⑦ $a\cdot b=\frac{1}{4}(\|a+b\|^2-\|a-b\|^2)$ (Polarization identity)

Proof of 5 :

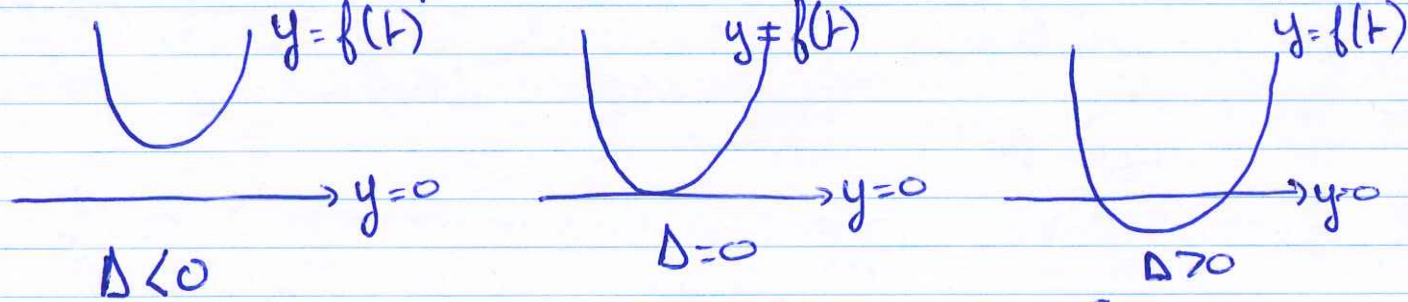
For $t \in \mathbb{R}$ we set $f(t) = \|a + tb\|^2$

then $f(t) = \|b\|^2 t^2 + 2(a \cdot b)\, t + \|a\|^2$

First case : $\|b\| = 0$ and then $b = 0$ and the result is obvious

Second case : $\|b\| \neq 0$ and then f is a quadratic polynomial with positive leading coefficient.

We have the following possibilities :



not possible since $f(t) \geq 0$

Hence $\Delta \leq 0$, but $\Delta = 4(a \cdot b)^2 - 4\|a\|^2\|b\|^2$

so $(a \cdot b)^2 \leq \|a\|^2 \|b\|^2$

and $|(a \cdot b)| \leq \|a\| \, \|b\|$ $\square$

Proof of 4 :

$$\|a+b\|^2 = \|a\|^2 + 2(a \cdot b) + \|b\|^2$$
$$\leq \|a\|^2 + 2|a \cdot b| + \|b\|^2$$
$$\leq \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2$$
$$= (\|a\| + \|b\|)^2$$

thus $\|a+b\| \leq \|a\| + \|b\|$ $\square$