(1) Assume that $f : U \to \mathbb{R}$ is not C^0 at $a \in U$ then

$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in U, |x-a| < \delta$ and $|f(x) - f(a)| \geqslant \varepsilon$

Let $\varepsilon > 0$ as above.

For $n \in \mathbb{N}_{>0}$, take $\delta = \frac{1}{n}$, then $\exists x_n \in U$ s.t. $\begin{cases} |x_n - a| < \frac{1}{n} & (1) \\ |f(x_n) - f(a)| \geqslant \varepsilon & (2) \end{cases}$

By (1) $\lim\limits_{n \to \infty} x_n = a$

Now $y_n = f(x_n) \in f(U) \subset \overline{f(U)}$ compact so

$\exists$ a subsequence $y_{\sigma(n)} = f(x_{\sigma(n)})$ s.t.

$L := \lim\limits_{n \to \infty} y_{\sigma(n)} \in \overline{f(U)}$

Define $u_n = \left(x_{\sigma(n)}, \overset{f(x_{\sigma(n)})}{\overset{\shortparallel}{y_{\sigma(n)}}}\right)$

then $u_n \xrightarrow[n \to \infty]{} (a, L)$

and $u_n \in \Gamma_f$ closed hence $(a, L) \in \Gamma_f$

ie $L = f(a)$

so $f(x_{\sigma(n)}) \xrightarrow[n \to \infty]{} f(a)$

$\exists N$ s.t. $|f(x_{\sigma(N)}) - f(a)| < \varepsilon$

Contradiction with (2)

② $f(x,y) = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Method 1 : notice either that

ⓐ $^t\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \cdot \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} = I_{2,2}$

ie it is an orthogonal matrix

ⓑ it is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2+b^2=1$

ie it is a rotation matrix

so $\|f(x,y)\| = \|(x,y)\|$

then by induction $\|f^n(p)\| = \|p\|$

so $(f^n(p))$ is bounded and admits a CV subsequence

Method 2 : $(3x+4y)^2 + (4x-3y)^2 = 25x^2+25y^2$

Hence $\|f(x,y)\| = \|(x,y)\|$

so by induction $(f^n(p))$ is bounded

and admit a CV subsequence

③ • f is C^0

• $f(0,0) = 0$ with $(0,0) \in B(\vec{0}, r)$

• $f(2,3) = 5^2 + 0 = 25$ with $(2,3) \in B(\vec{0}, r)$

• $B(\vec{0}, r)$ is path-connected

• $0 < 20 < 25$

Hence by the IVT, $\exists\, \vec{a} \in B(\vec{0}, r)$ s.t. $f(\vec{a}) = 20$

④ Let $p \in \mathbb{R}^2$, $t \in \mathbb{R}$

define $\varphi(s) = f(p + stv)$ for $s \in [0,1]$

then • φ is C^0 on $[0,1]$

• φ is diff on $(0,1)$

so by the MVT, $\exists\, s_0 \in (0,1)$ s.t.

$$\varphi(1) - \varphi(0) = \varphi'(s_0)(1-0)$$

but $\varphi(1) = f(p+tv)$, $\varphi(0) = f(p)$

and by the chain rule $\varphi'(s_0) = d_{s_0}\varphi(1) = d_{p+s_0tv} f(tv)$

$= t\, d_{p+s_0tv} f(v)$

$= t\, \partial_v f(p + s_0 tv) = 0$

$\varphi(s) = f(\alpha(s))$ where $\alpha(s) = p + stv$

Hence $\varphi'(s_0) = \nabla f(\alpha(s_0)) \cdot \underbrace{(tv)}_{\alpha'(s_0)}$

$= t\, \nabla f(\alpha(s_0)) \cdot v = t\, \partial_v f(\alpha(s_0))$

$= 0$

if you don't like differentials

Hence $f(p+tv) - f(p) = 0$

ie $f(p+tv) = f(p)$

(5) $f(0,\pi/2) = \frac{\pi}{2}$

$\frac{\partial f}{\partial x}(0,\pi/2) = 0$

$\frac{\partial f}{\partial y}(0,\pi/2) = 1$

$\frac{\partial^2 f}{\partial x^2}(0,\pi/2) = -\pi/2$

$\frac{\partial^2 f}{\partial y^2}(0,\pi/2) = 0$

$$\frac{\partial^2 f}{\partial y \partial x}(0,\pi/2) \underset{\substack{\uparrow \\ f\, C^2 \text{ by Clairaut}}}{=} \frac{\partial^2 f}{\partial x \partial y}(0,\pi/2) = -1$$

hence
$$P_{(0,\pi/2),2}(h,k) = f(0,\pi/2) + \frac{\partial f}{\partial x}(0,\pi/2)h + \frac{\partial f}{\partial y}(0,\pi/2)k + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,\pi/2)h^2 + \frac{\partial^2 f}{\partial y^2}(0,\pi/2)k^2 + \frac{\partial^2 f}{\partial x \partial y}(0,\pi/2)hk + \frac{\partial^2 f}{\partial y \partial x}(\cdots)kh\right)$$

$$= \frac{\pi}{2} + 0h + 1k + \frac{1}{2}\left(-\frac{\pi}{2}h^2 + 0k^2 - 2hk\right)$$

$$= \frac{\pi}{2} + k - \frac{\pi}{4}h^2 - hk$$

or
$$P_{(0,\pi/2),2}(h,k) = f(0,\pi/2) + \nabla f(0,\pi/2)\begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2}\begin{pmatrix} h \\ k \end{pmatrix}^t Hf(0,\pi/2)\begin{pmatrix} h \\ k \end{pmatrix}$$

$$= \frac{\pi}{2} + \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2}\begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} -\pi/2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

⑥ f is differentiable on $\mathbb{R}^2$

$\nabla f(x,y) = (4x-4y-12, -4x+y^2)$

$$\begin{cases} 4x-4y-12=0 \\ 4x=y^2 \end{cases} \Leftrightarrow \begin{cases} y^2-4y-12=0 \\ 4x=y^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} (y+2)(y-6)=0 \\ 4x=y^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=9 \\ y=6 \end{cases} \text{ or } \begin{cases} x=1 \\ y=-2 \end{cases}$$

So we have 2 critical points : $(9,6)$, $(1,-2)$

$$H_f(x,y) = \begin{pmatrix} 4 & -4 \\ -4 & 2y \end{pmatrix}$$

- $H_f(9,6) = \begin{pmatrix} 4 & -4 \\ -4 & 12 \end{pmatrix}$

$4 \times 12 - (-4)^2 = 32 > 0$ local extremum

$4 > 0$: local min

$(9,6)$ local min

- $H_f(1,-2) = \begin{pmatrix} 4 & -4 \\ -4 & -4 \end{pmatrix}$: saddle point

$4 \times (-4) - (-4)^2 = -32 < 0$

$(1,-2)$ saddle point

(7) $f(x,y,z) = x$

(I went very fast so double check my computations)

• $S: \begin{cases} 5x+4y+6z=0 & (1) \\ x^2+2y^2+3z^2 = 9 & (2) \end{cases}$

$\Rightarrow$ S is closed has the intersection of 2 closed sets (each one is the preimage of a singleton closed by a C^0 function)

$\Rightarrow$ S is bounded since the ellipsoid is too

$\Rightarrow \boxed{S \text{ is compact}}$

f is C^0 on $S \underset{(EVT)}{\Rightarrow} f$ has a max and a min on S

• let $g(x,y,z) = 5x+4y+6z$ $\rightarrow C^1$ $\quad h(x,y,z) = x^2+2y^2+3z^2$ $\rightarrow C^1$

$\nabla g = (5,4,6) \quad \nabla h = (2x, 4y, 6z)$

$\nabla h = \lambda \nabla g \Rightarrow (x,y,z) = (\frac{5\lambda}{2}, \lambda, \lambda) \notin S$

so ∇h & ∇g are lin ind on S

g, h, f is C^1 so if $\vec{a}$ is a local extremum of f on S

$\nabla f(a) = \lambda \nabla g(a) + \mu \nabla h(a) \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 5 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2x \\ 4y \\ 6z \end{pmatrix}$

$\mu = 0 \Rightarrow \lambda = 0$ contradict°

so $y = z = -\frac{\lambda}{\mu}$, $x = \frac{1-5\lambda}{2\mu}$

$\left.\begin{array}{l} (1) \Rightarrow \lambda = 1/9 \\ (2) \Rightarrow \mu = \pm 1/9 \end{array}\right\} \Rightarrow (x,y,z) = (2,-1,-1)$ or $(x,y,z) = (-2,1,1)$

$f(2,-1,-1) = 2$ max

$f(-2,1,1) = -2$ min

(8) g is C^1 and $\nabla g(0) = (1,1,-2)$ so $\frac{\partial g}{\partial z} \neq 0$

Hence by the IFT $g = g(0)$ is locally a graph ~~around~~ around 0

$$z = \varphi(x,y), \quad \varphi \ C^1$$

define $\gamma(t) = (t, t, \varphi(t,t))$

then $\gamma \in \{g = g(0)\}$ by construction of φ

$$\gamma'(0) = \begin{pmatrix} 1 \\ 1 \\ \nabla\varphi\binom{0}{0}\cdot\binom{1}{1} \end{pmatrix}$$

but $\nabla\varphi\binom{0}{0} = -\dfrac{\binom{1}{1}}{-2} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

and ~~$\nabla\varphi\binom{0}{0} = \binom{1/2}{1/2}\cdot\binom{1}{1} = 1$~~ $\nabla\varphi\binom{0}{0}\cdot\binom{1}{1}$
$= \binom{1/2}{1/2}\cdot\binom{1}{1} = 1$

and $\gamma'(0) = (1,1,1)$

So we constructed a C^1 arc $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$

s.t. $\forall r, \gamma(r) \in \{g = g(0)\}$

and $\gamma(0) = (0,0,\varphi(0,0)) = (0,0,0)$

and $\gamma'(0) = (1,1,1)$

$\Rightarrow (1,1,1)$ is tangent to $\{g = g(0)\}$ at $(0,0,0)$

9. • On $(1,\infty)$:

$$f'(x) = -\frac{1}{2x^{3/2}} \Rightarrow |f'(x)| < 1/2 \text{ on } (1,\infty)$$

By the MVT, f is Lipschitz hence UC

• On $(0,1)$:

$$\lim_{x\to 0^+} \frac{1}{x} \text{ DNE} \Rightarrow f \text{ is not UC}$$

10. I misread the question, so here is the good solution

Using cylindrical coordinates:

$$\mathrm{Vol}(S) = \iiint_S 1 = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{r^2\cos^2\theta + r\sin\theta} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\int_0^1 (r^2\cos^2\theta + r\sin\theta)r\mathrm{d}r\mathrm{d}\theta \quad = ...$$

11) • Comment : $f(x) = \frac{1}{\|x\|^6}$, $6 > 3$ so f is integrable

• Since f is C^0 on $\mathbb{R}^2 \setminus \bar{B}(0,2)$ and non-negative

$$\iiint_{\mathbb{R}^3 \setminus B(0,2)} f \underset{\uparrow}{=} \lim_{k \to +\infty} \iiint_{2 \leq \|x\| \leq k} \frac{1}{\|x\|^6}$$

⚠ $f \geq 0$ so doesn't depend on the exhaustion

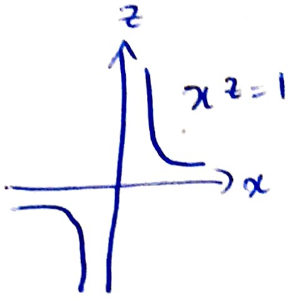
(spherical coord) $$= \lim_{k \to +\infty} \int_0^{\pi} \int_0^{2\pi} \int_2^k \frac{1}{r^6} r^2 \sin\varphi \, dr \, d\theta \, d\varphi$$

$$= 2\pi \cdot \left[-\cos\varphi\right]_0^{\pi} \cdot \left[-\frac{r^{-3}}{3}\right]_2^{\infty}$$

$$= \frac{4\pi}{24} = \frac{\pi}{6}$$

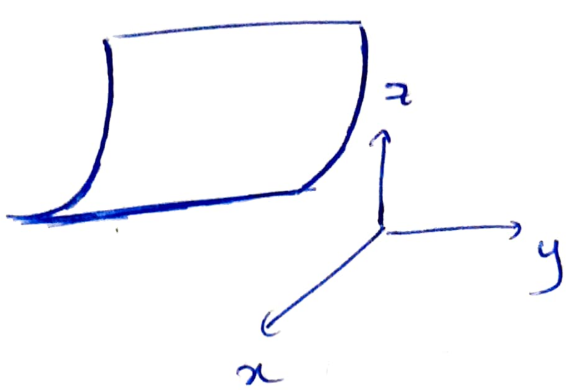
12) Just use the formula $\operatorname{div}(fG) = f \operatorname{div} G + \nabla f \cdot G$

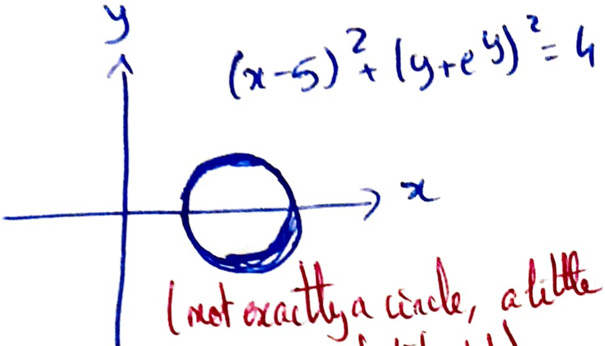
(13) $C = \{ xz = 1,\ (x-5)^2 + (y+e^y)^2 = 4 \}$



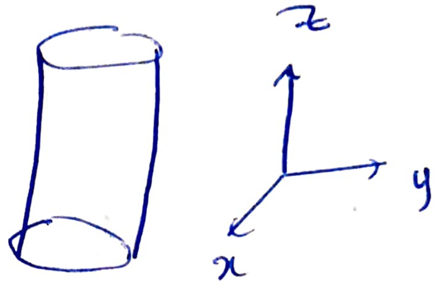
so

$x > 0,\ xz = 1$

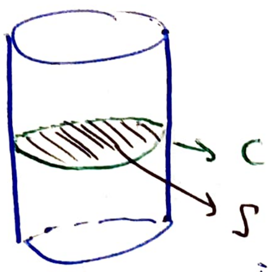




(not exactly a circle, a little bit distorded)



So



$S = \{ xz = 1,\ (x-5)^2 + (y+e^y)^2 \leq 4 \}$

We know that S is ~~orthogonal to~~ included in the level set $xz = 1$
so $\vec{n}$ is colinear to $\nabla(xz) = (z, 0, x)$ since the gradient is ~~always~~ orthogonal to the level sets

ie : $\vec{n} = \lambda(x,y,z) \cdot (z, 0, x)$

Stokes : $\displaystyle\int_C \vec{F} \cdot d\vec{x} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} = \iint_S \lambda \begin{pmatrix} -x \\ 0 \\ z \end{pmatrix} \cdot \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$

$= 0$

(14) See poincare.pdf

and my review for 5.5 → 5.7

(15) We assume that $G(x)\cdot x > 0$ for $x \in \partial B(0,1)$

Assume by contradiction that $G = \text{curl} F$ for F C^2

<u>Method 1</u>: $\displaystyle\iint_{\partial B(0,1)} \vec{G}\cdot\vec{n} = \iint_{\partial B(0,1)} \text{curl}\vec{F}\cdot\vec{n} \underset{\uparrow}{=} 0$

Stokes since $\partial B(0,1)$ is closed

$\vee$
0 since $\vec{n}(x,y,z) = \pm(x,y,z)$ on $\partial B(0,1)$

(or < 0 depending on the orientation)

<u>Method 2</u>: $\displaystyle\iint_{\partial B(0,1)} \vec{G}\cdot\vec{n} = \iiint_{B(0,1)} \text{div}\, G = \iiint_{B} \underbrace{\text{div}(\text{curl} F)}_{=0} = 0$

since F is C^2

>0 or <0

depending on the orientation

Method 1 always works tho

$\boxed{\text{not method 2}}$ it doesn't apply if

G is not defined on $B(0,1)$

eg: $G = \dfrac{x}{\|x\|^3}$ not defined at 0