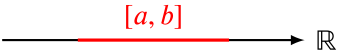
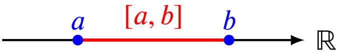
Relative boundaries in Stokes' theorem

April 2nd, 2020

Let $C = [a, b] \subset \mathbb{R}$. What is ∂C, the boundary of C?



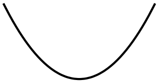
Let $C = [a, b] \subset \mathbb{R}$. What is ∂C, the boundary of C?



Of course, $\partial C = \overline{C} \setminus C^\circ = \{a, b\}$.

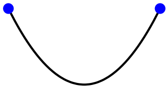
A review from the first lecture

Let $C = \left\{ \left(t, t^2\right) \ : \ t \in [-1, 1] \right\} \subset \mathbb{R}^2$.
What is ∂C, the boundary of C?



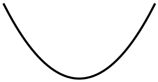
A review from the first lecture

Let $C = \left\{ \left(t, t^2\right) \ : \ t \in [-1, 1] \right\} \subset \mathbb{R}^2$.
What is ∂C, the boundary of C?



Is it $\partial C = \{(-1, 1), (1, 1)\}$?

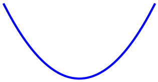
Let $C = \left\{ \left(t, t^2\right) \;:\; t \in [-1, 1] \right\} \subset \mathbb{R}^2$.
What is ∂C, the boundary of C?



Is it $\partial C = \{(-1, 1), (1, 1)\}$?
No, remember that $\partial C = \overline{C} \setminus C^\circ$.

A review from the first lecture

Let $C = \left\{ \left(t, t^2\right) \ : \ t \in [-1, 1] \right\} \subset \mathbb{R}^2$.
What is ∂C, the boundary of C?



Is it $\partial C = \{(-1, 1), (1, 1)\}$?
No, remember that $\partial C = \overline{C} \setminus C^\circ$.
So $\partial C = C$.

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying "the boundary of C", because

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying “the boundary of C”, because

- Intuitively, it is or, at least, it should be…

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

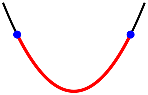
Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying “the boundary of C”, because

- Intuitively, it is or, at least, it should be…
  But that’s a very debatable argument…

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying "the boundary of C", because

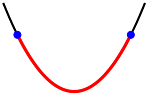
- Intuitively, it is or, at least, it should be…
  But that's a very debatable argument…
- Actually, it is the boundary of C if you don't see C as a subset embedded in $\mathbb{R}^2$ but instead as a subset of $X = \{(t, t^2) \,:\, t \in \mathbb{R}\}$ for some topology.



Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying "the boundary of C", because

- Intuitively, it is or, at least, it should be…
  But that's a very debatable argument…
- Actually, it is the boundary of C if you don't see C as a subset embedded in $\mathbb{R}^2$ but instead as a subset of $X = \{(t, t^2) \;:\; t \in \mathbb{R}\}$ for some topology.

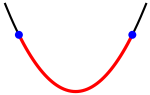


- It is also possible to see C as an intrinsec object independently of any embedding and then, by definition, the boundary of C as an abstract manifold is made of the endpoints.
  That's why we write $\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega$ for the general Stokes' theorem.

Thus the *topological boundary of a non-closed curve as a subset of* $\mathbb{R}^2$ is not made of the endpoints.

Nevertheless, when talking about the Gradient Theorem, it is common to denote the endpoints by saying "the boundary of C", because

- Intuitively, it is or, at least, it should be…
  But that's a very debatable argument…
- Actually, it is the boundary of C if you don't see C as a subset embedded in $\mathbb{R}^2$ but instead as a subset of $X = \{(t, t^2) \ : \ t \in \mathbb{R}\}$ for some topology.



- It is also possible to see C as an intrinsec object independently of any embedding and then, by definition, the boundary of C as an abstract manifold is made of the endpoints.
  That's why we write $\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega$ for the general Stokes' theorem.

You have to be very careful about the meaning of the boundary of a set.
In MAT237, you should assume we mean the boundary for C as a subset of $\mathbb{R}^2$ as in the first chapter.

Can we start talking about Stokes'?

Ok, but isn't today's lecture about (Kelvin-)Stokes' theorem?

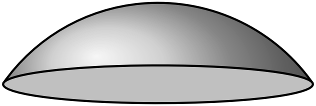
Can we start talking about Stokes'?

Ok, but isn't today's lecture about (Kelvin-)Stokes' theorem?
Yes, but the Gradient Theorem is to the FTC what Stokes' theorem is to Green's theorem.
So we have to be careful about what we mean by boundary.

Can we start talking about Stokes'?

Ok, but isn't today's lecture about (Kelvin-)Stokes' theorem?
Yes, but the Gradient Theorem is to the FTC what Stokes' theorem is to Green's theorem.
So we have to be careful about what we mean by boundary.

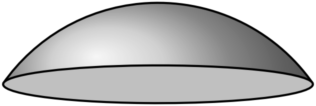
Let $S = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\right\}$. What is ∂S?



Can we start talking about Stokes'?

Ok, but isn't today's lecture about (Kelvin-)Stokes' theorem?
Yes, but the Gradient Theorem is to the FTC what Stokes' theorem is to Green's theorem.
So we have to be careful about what we mean by boundary.

Let $S = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\right\}$. What is ∂S?

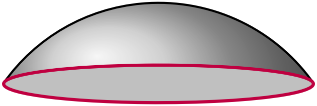


Again, $\partial S = \overline{S} \setminus S^\circ = S$.

Can we start talking about Stokes'?

Ok, but isn't today's lecture about (Kelvin-)Stokes' theorem?
Yes, but the Gradient Theorem is to the FTC what Stokes' theorem is to Green's theorem.
So we have to be careful about what we mean by boundary.

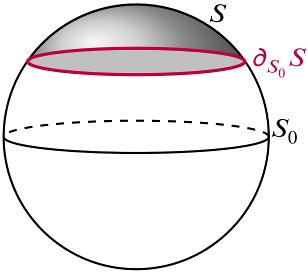
Let $S = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\right\}$. What is ∂S?



Again, $\partial S = \overline{S} \setminus S^\circ = S$.

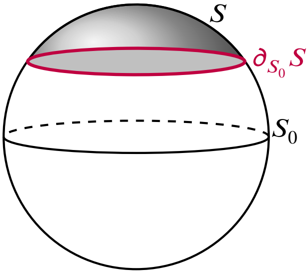
But that's not what we want for Stokes' theorem:
we want the circle in purple!

In order to be formal/precise, I'll use the following setup in the lecture:
Set $S_0 = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\right\}$ and
$S = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\right\}$.



Then we define the ***relative boundary of S with respect to S_0*** by
$\partial_{S_0} S = \left\{x \in S_0 \ : \ \forall \varepsilon > 0, \ B(x, \varepsilon) \cap S \neq \varnothing \text{ and } B(x, \varepsilon) \cap (S_0 \setminus S) \neq \varnothing\right\}$.

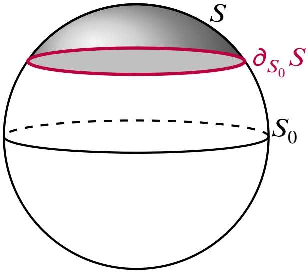
In order to be formal/precise, I'll use the following setup in the lecture:
Set $S_0 = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\right\}$ and
$S = \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\right\}$.



Then we define the ***relative boundary of S with respect to S_0*** by
$\partial_{S_0} S = \left\{x \in S_0 \ : \ \forall \varepsilon > 0, \ B(x, \varepsilon) \cap S \neq \varnothing \text{ and } B(x, \varepsilon) \cap (S_0 \setminus S) \neq \varnothing\right\}$.

Notice that I took the complement in S_0, not in $\mathbb{R}^3$.

In order to be formal/precise, I'll use the following setup in the lecture:
Set $S_0 = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\}$ and
$S = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ z \geq 0.6\}$.



Then we define the ***relative boundary of S with respect to S_0*** by
$\partial_{S_0} S = \{x \in S_0 \ : \ \forall \varepsilon > 0, \ B(x, \varepsilon) \cap S \neq \emptyset$ and $B(x, \varepsilon) \cap (S_0 \setminus S) \neq \emptyset\}$.

Notice that I took the complement in S_0, not in $\mathbb{R}^3$.

Beware: in practice, we are less careful while stating Stokes' theorem and we usually simply say *boundary* and drop the S_0 to simply write ∂S.
You will have to rely on the context.