Vector potentials (or vector fields that are curls)

Here is another special case of Poincaré Lemma:

Theorem: $\mathcal{U} \subset \mathbb{R}^3$ open, $F : \mathcal{U} \to \mathbb{R}^3$ C^1

① If $F = \operatorname{curl} G$ for $G : \mathcal{U} \to \mathbb{R}^3$ C^2 then $\operatorname{div} F = 0$

② The converse is true when $\mathcal{U}$ is star-shaped.
If $\mathcal{U}$ is star-shaped and $\operatorname{div} F = 0$ then there exists $G : \mathcal{U} \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

We say that G is a vector potential of F

Δ ① $\operatorname{div} F = \operatorname{div}(\operatorname{curl} G) = 0$

② Since $\mathcal{U}$ is star-shaped, $\exists p_0 \in \mathcal{U}$ s.t. $\forall q \in \mathcal{U}$ the line segment from p_0 to q is in $\mathcal{U}$.

For $q = (x, y, z) \in \mathcal{U}$ we set: we apply the $\int$ componentwise

$$G(q) = \int_0^1 F(\underbrace{(1-t)p_0 + tq}_{\in \mathcal{U} \text{ so well defined}}) \times (t(q - p_0))\, dt$$

Using the theorem to differentiate under the integral, we get that G is C^2 and that $\operatorname{curl} G = \int_0^1 \operatorname{curl}(\ \times\ )\, dt$

where the curl are w.r.t. x, y, z of course

To simplify the notations, I set:

$$\tilde{F}(x,y,z) = F((1-t)P_0 + tq) \qquad \text{(remember } q=(x,y,z))$$

$$= F((1-t)P_{0,x} + tx, (1-t)P_{0,y} + ty, (1-t)P_{0,z} + tz)$$

$$= F(P_0 + tr(x,y,z))$$

and

$$r(x,y,z) = q - P_0 = (x - P_{0,x}, y - P_{0,y}, z - P_{0,z})$$

then we use the formula $\operatorname{curl}(F \times G) = (G \cdot \nabla) F + (\operatorname{div} G) F - (F \cdot \nabla) G - (\operatorname{div} F) G$

(Be carefule, I've just realized that the formula in the online text book is false)

$$\operatorname{curl}(\tilde{F} \times (tr)) = \underset{①}{(tr \cdot \nabla)\tilde{F}} + \underset{②}{(\operatorname{div}(tr))\tilde{F}} - \underset{③}{(\tilde{F} \cdot \nabla)(tr)} - \underset{④}{\operatorname{div}(\tilde{F})G}$$

① $(tr \cdot \nabla)\tilde{F} = t\left((x - P_{0,x})\frac{\partial}{\partial x} + (y - P_{0,y})\frac{\partial}{\partial y} + (z - P_{0,z})\frac{\partial}{\partial z}\right)\tilde{F}$

$$= t^2 \frac{d}{dt}(F((1-t)P_0 + tq))$$

② $\operatorname{div}(tr)\tilde{F} = 3t\, F((1-t)P_0 + tq)$

③ $(\tilde{F} \cdot \nabla)(tr) = t\, F((1-t)P_0 + tq)$

④ $\operatorname{div}(\tilde{F}) = t\frac{\partial F_1}{\partial x}(-) + t\frac{\partial F_2}{\partial y}(-) + t\frac{\partial F_3}{\partial z}(-) = t\,\overbrace{\operatorname{div} F((1-t)P_0 + tq)}^{=0} = 0$

Hence $\operatorname{curl} G(q) = \int_0^1 2t\, F((1-t)P_0 + tq) + t^2 \frac{d}{dt}(F((1-t)P_0 + tq))\, dt$

$$= \int_0^1 \frac{d}{dt}\left(t^2 F((1-t)P_0 + tq)\right) dt$$

$$= \left[t^2 F((1-t)P_0 + tq)\right]_{t=0}^{t=1}$$

$$= F(q)$$

$\square$

Notice that the above proof gives a formula to find a suitable G:

$$G(x,y,z) = \int_0^1 F((1-t)P_0 + t(x,y,z)) \times (t((x,y,z)-P_0))\, dt$$

where $P_0 \in U$ s.t. $\forall q \in U$ the line segment from P_0 to q is included in U

Quite often, in practice, U is centred at O so that $P_0 = \vec{O}$ works

The above theorem fails with no assumption on the domain

Define $F(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$ on $U = \mathbb{R}^3 \setminus \{\vec{0}\}$

then $\operatorname{div} F = 0$ but there is no $G : U \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

Indeed, otherwise we would have $\displaystyle\iint_{\substack{\{x^2+y^2+z^2=1\} \\ \text{outward}}} \vec{F}\cdot\vec{n} = 0$

but $\displaystyle\iint \vec{F}\cdot\vec{n} = 4\pi \neq 0$