Conservative vector fields

Theorem: $\mathcal{U} \subset \mathbb{R}^n$ open, $n \geq 2$, $F : \mathcal{U} \to \mathbb{R}^n$ C^0

TFAE: ① $\exists f : \mathcal{U} \to \mathbb{R}$ C^1 s.t. $F = \nabla f$

② $\int_C \vec{F} \cdot d\vec{x} = 0$ for any closed piecewise smooth oriented curve C in $\mathcal{U}$

③ $\int_{C_1} \vec{F} \cdot d\vec{x} = \int_{C_2} \vec{F} \cdot d\vec{x}$ "path-independence"

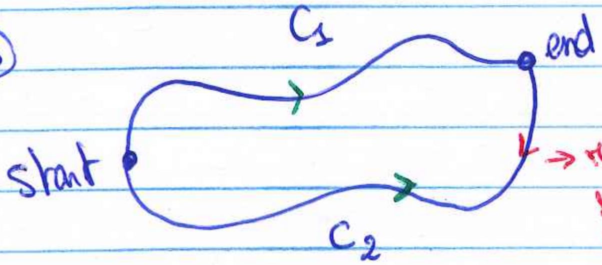
for any two oriented piecewise smooth curves in $\mathcal{U}$ with same start point and same end point

Def: In the above case, we say that $\vec{F}$ is conservative

Δ ① $\Rightarrow$ ② by the Gradient theorem:

$$\int_C \vec{F} \cdot dx = f(\text{endpoint}) - f(\text{startpoint}) = f(p) - f(p) = 0$$

② $\Rightarrow$ ③



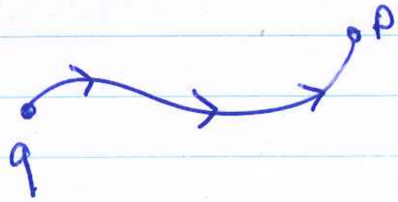
→ reverse orientation for C_2
I will use the notation $(-C_2)$

$C = C_1 \cup (-C_2)$

then $0 = \int_{C_1} \vec{F} \cdot d\vec{x} + \int_{-C_2} \vec{F} \cdot d\vec{x} = \int_{C_1} \vec{F} d\vec{x} - \int_{C_2} \vec{F} \cdot d\vec{x}$

③ ⇒ ① Pick $q \in \mathcal{U}$ and for $p \in \mathcal{U}$ set

$$f(p) = \int_C \vec{F} \cdot d\vec{x}$$ where C is any curve from q to p (the integral doesn't depend on the choice by assumption)



$$f(p+te_i) = \int_{q \to p+tv} \vec{F} \cdot d\vec{x} = \int_{q \to p} \vec{F} \cdot d\vec{x} + \int_{p \to p+te_i} \vec{F} \cdot d\vec{x}$$

$$= f(p) + \int_{p,\, p+te_i} \vec{F} \cdot d\vec{x}$$

So $$\frac{f(p+te_i) - f(p)}{t} = \frac{1}{t} \int_0^t F(p+se_i) \cdot e_i \, ds$$

$$= \frac{1}{t} \int_0^t F_i(p+se_i)\, ds \xrightarrow[t \to 0]{} F_i(p)$$

So $\nabla f = F$ C^0, and f C^1 $\square$

We are now going to give several special case $(n=2,3)$ of a very general result called "Poincaré lemma".

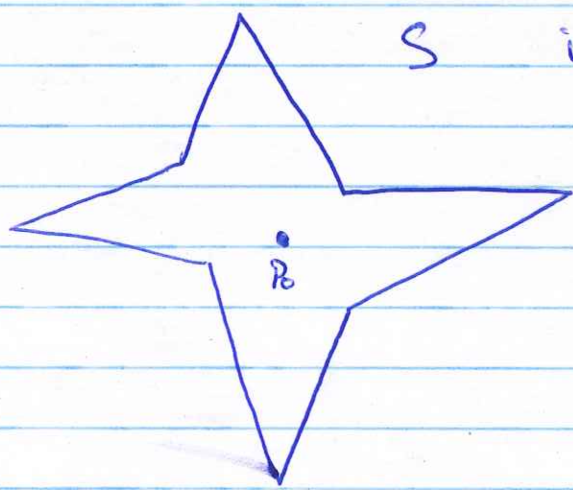
Def: We say that $S \subset \mathbb{R}^n$ is star-shaped if

$$\exists p_0 \in S \text{ st. } \forall p \in S \text{ the line segment from } p_0 \text{ to } p \text{ lies in } S$$

⚠ Obviously "$\neq \emptyset$ + convex $\Rightarrow$ star-shaped" but the converse is false (here we have p_0 a fixed start point)

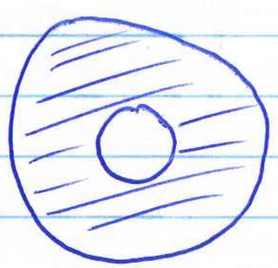
$\emptyset$ is convex not star shaped

Ex:



S is star shaped but not convex

Ex:



is not star shaped

$(n=2)$

Theorem: $\mathcal{U} \subset \mathbb{R}^2$ open, $F : \mathcal{U} \to \mathbb{R}^2$ C^1

① If $\vec{F} = \nabla f$ for $f : \mathcal{U} \to \mathbb{R}$ C^2 then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

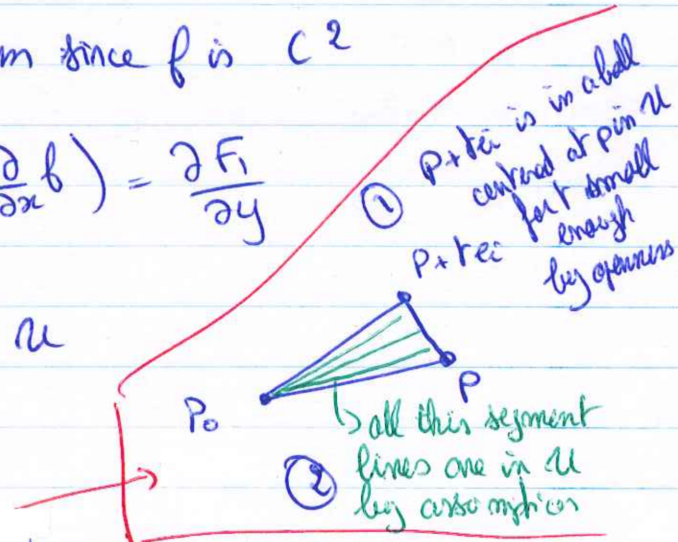
↳ "$\vec{F}$ is conservative"

② The converse is true when $\mathcal{U}$ is star-shaped:
If $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ on $\mathcal{U}$ star-shaped then $\exists f : \mathcal{U} \to \mathbb{R}$ C^2

such that $\vec{F} = \nabla f$

Δ ① it's simply by Clairaut's theorem since f is C^2

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial F_1}{\partial y}$$



② $\exists p_0 \in \mathcal{U}$ st. $\forall p \in \mathcal{U}$, $[p_0, p] \subset \mathcal{U}$
Let $f(p) = \displaystyle\int_{[p_0,p]} \vec{F} \cdot d\vec{x}$

For $t \geq 0$ small enough the triangle $P_0 \longrightarrow p + te_i \longrightarrow P \longrightarrow P_0$ and its interior lies in $\mathcal{U}$

So
$$0 = \iint_T \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{=0} \overset{\text{Green}}{=} \int_{\partial T} \vec{F} \cdot d\vec{x} = \int_{P_0 \to P} \vec{F} \cdot d\vec{x} + \int_{P \to p + te_i} \vec{F} \cdot d\vec{x} + \int_{p+te_i \to P_0} \vec{F}\, d\vec{x}$$

$$= f(P) + \int_0^t F(p + se_i) \cdot e_i\, ds - f(P + te_i)$$

So
$$\frac{f(p+te_i) - f(P)}{t} = \frac{1}{t} \int_0^t F_i(p + se_i)\, ds \xrightarrow[t \to 0]{} F_i(P)$$

□

The star-shaped assumption is important

Ex: Define $F(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$ on $\mathbb{R}^2 \setminus \{0\}$

then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ but F is $\boxed{\text{not}}$ conservative.

Indeed, take C the unit circle counterclockwised oriented then

$$\int_C \vec{F} \cdot dx = \int_0^{2\pi} \underbrace{\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}}_{F(\sigma(t))} \cdot \underbrace{\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}_{\sigma'(t),\ \sigma = (\cos t, \sin t)} dt$$

$$= \int_0^{2\pi} -1 \, dt = -2\pi \neq 0$$

whereas C is closed!

Theorem. $(n=3)$ $U \subset \mathbb{R}^3$ open, $F: U \to \mathbb{R}^3$ C^1

1. If $\vec{F} = \nabla f$ for $f: U \to \mathbb{R}$ C^2 then $\operatorname{curl} \vec{F} = \vec{0}$
2. The converse hold on a starshaped domain: If U is starshaped and $\operatorname{curl} \vec{F} = \vec{0}$ then $\vec{F} = \nabla f$ for $f: U \to \mathbb{R}$ C^2

Δ Same proof as before by replacing Green by Stokes $\square$

Assume that $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field s.t.
$\operatorname{curl} \vec{F} = \vec{0}$

How can we find a potential $f: \mathbb{R}^3 \to \mathbb{R}$ C^2 s.t. $\vec{F} = \nabla f$?

We just follow the first proof of this section!

Take $q = (a,b,c)$ and $p = (x,y,z)$

We know that $f(p) = \int_{C_{q \to p}} \vec{F} \cdot d\vec{x}$ works

$\hookrightarrow$ doesn't depend on the curve from q to p!

We take the following segment line:

$$(a,b,c) \xrightarrow{L_1} (x,b,c) \xrightarrow{L_2} (x,y,c) \xrightarrow{L_3} (x,y,z)$$

$\sigma_1(t) = (t,b,c)$, $t \in [a,x]$ $\quad$ $\sigma_2(t) = (x,t,c)$, $t \in [b,y]$ $\quad$ $\sigma_3(t) = (x,y,t)$, $t \in [c,z]$

Then $f(p) = \int_{q \to p} \vec{F} \cdot d\vec{x} = \int_{L_1} \vec{F} \cdot d\vec{x} + \int_{L_2} \vec{F} \cdot d\vec{x} + \int_{L_3} \vec{F} \cdot d\vec{x}$

$$\boxed{f(p) = \int_a^x F_1(t,b,c)\,dt + \int_b^y F_2(x,t,c)\,dt + \int_c^z F_3(x,y,t)\,dt}$$