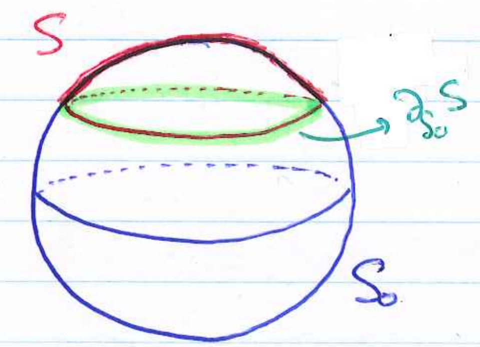
# Stokes theorem (Also known as Kelvin-Stokes or curl theorem)

**Setup:**
- $S_0 \subset \mathbb{R}^3$ a smooth surface
- $S$ a surface lying in $S_0$
- We denote the relative boundary of $S$ in $S_0$ by $\partial_{S_0} S$ (we assume it is a curve or $\emptyset$)



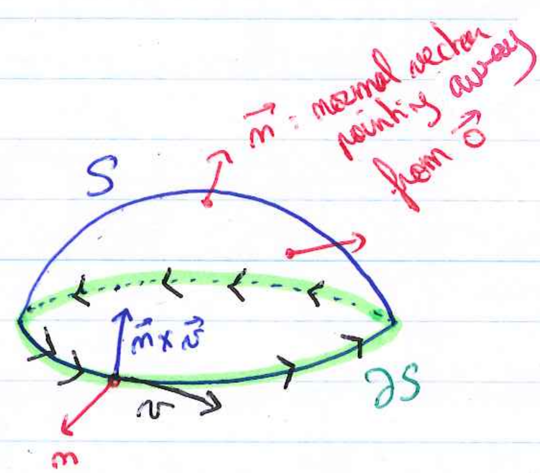

(We often drop the $S_0$ and simply write $\partial S$, but be careful $\partial_{S_0} S \neq \partial S$)

**Comment:** by "relative" we mean that we forget $\mathbb{R}^3$ and see the ambiant space as $S_0$ for some topology.
I won't give details because it outreaches MAT237 but that's a very intuitive notion.

If you want a formal definition for $\partial_{S_0} S$:

$$\partial_{S_0} S = \{ x \in S_0 : \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset \text{ and } B(x,\varepsilon) \cap (S_0 \setminus S) \neq \emptyset \}$$

- We assume that $S$ is oriented by $\vec{n} : S \to \mathbb{R}^3$ a $C^0$ normal unit vector field
- We assume that $\partial_{S_0} S$ is positively oriented: if $\vec{v}$ gives the orientation of $\partial_{S_0} S$ then $\vec{n} \times \vec{v}$ points to $S$



**Theorem** (Stokes, Kelvin-Stokes or curl)

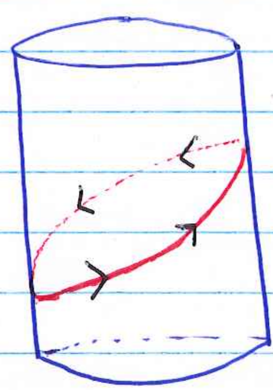
$F : \mathcal{U} \longrightarrow \mathbb{R}^3$ $C^1$, $\mathcal{U} \subset \mathbb{R}^3$ open, $S \subset \mathcal{U}$, then

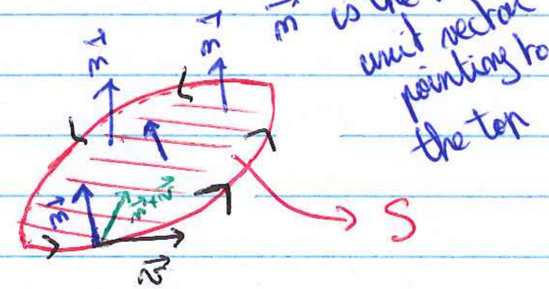
$$\int_{\partial_{S_0} S} \vec{F} \cdot d\vec{x} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{m}$$

**Comment**: that's a "3D" version of Green's theorem: $\to S_0 =$ plane
the latter is a special case when the surface is "flat"
(ie inside a plane)

**Comment**: that's (again!) a special case of the general Stokes theorem for differential forms.

**Ex**: We define $C$ as the ellipse obtained by intersecting the cylinder $x^2+y^2=1$ with the plane $y-z=0$ oriented counterclockwise as viewed from above.
Compute $\int_C (x-z)\,dx + (x+y)\,dy + (y+z)\,dz$





9.
$$\int_C (x-z)\,dx + (x+y)\,dy + (y+z)\,dz$$

$$= \iint_S \operatorname{curl} \vec{F} \cdot \vec{m} \quad \text{where } \vec{F}(x,y,z) = (x-z,\, x+y,\, y+z)$$

$$\operatorname{curl} \vec{F} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\vec{m}$ is the unit vector normal to $y-z=0$ pointing to the top

so
$$\vec{m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and $\operatorname{curl} \vec{F} \cdot \vec{m} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

so $\displaystyle\int_C \vec{F} \cdot d\vec{x} = \iint_S \sqrt{2}$

then notice that $S = \{ \underbrace{(r\cos\theta, r\sin\theta, r\sin\theta)}_{\sigma(r,\theta)} : r \in [0,1],\ \theta \in [-\pi,\pi] \}$

$$\partial_1 \sigma \times \partial_2 \sigma = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \sin\theta \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -r \\ r \end{pmatrix}$$

Hence $\displaystyle\int_C \vec{F} \cdot d\vec{x} = \int_{-\pi}^{\pi} \int_0^1 \sqrt{2}\, \| (0,-r,r) \| \, dr\, d\theta$

$$= 4\pi \int_0^1 r\, dr$$

$$= 2\pi.$$

In this corollary, I am not very formal.
It is possible to define properly what do I mean by "closed", but I think that's not necessary for our purpose.
You can believe your intuition: "closed" means that there is no "boundary" in the above sense, so the line integral is 0.

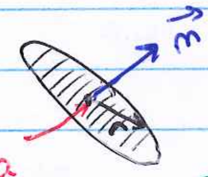
**Corollary:**

Assume that S is "closed"

then $\displaystyle\iint_S (\operatorname{curl}\vec F)\cdot\vec m = 0$

Stokes theorem allows to give a physics interpretation of curl.

Let $a\in\mathbb{R}^3$, $\vec m$ a unit vector and $D_r$ the disk centered at a normal to $\vec m$ and of radius r




intuitively the average goes to the exact value! we may prove it with a "MVT for $\int$" theorem for surfaces integrals

then $$(\operatorname{curl}\vec F(a))\cdot\vec m = \lim_{r\to0}\frac{1}{\pi r^2}\iint_{D_r}(\operatorname{curl}\vec F)\cdot\vec m$$

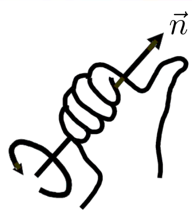
$$= \lim_{r\to0}\frac{1}{\pi r^2}\int_{C_r}\vec F\cdot d\vec x \qquad (*)$$

If $\vec F$ is a force field, then $\int_{C_r}\vec F\cdot d\vec x$ is the work of F on a particle moving along $C_r$:

So $\operatorname{curl}\vec F(a)\cdot\vec m > 0$ : the force pushes the particle counterclockwise

$< 0$ ———————— clockwise

By "counterclockwise", I mean that the force pushes the particle in the direction of the orientation of the boundary, as in the "right hand rule" on the right:

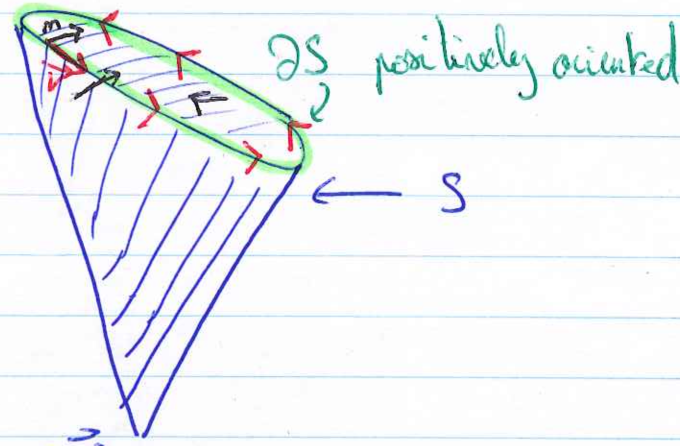



(Image from Wikipedia)

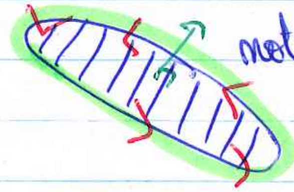
By the way, (*) doesn't depend on the coordinate system!

## Another example of usage:

Let $S$ be the part of the cone $z = \sqrt{x^2+y^2}$ below $x+z=1$
oriented by $\vec{n}$ going upward (in the inside)




we want to compute $\iint_S (\mathrm{curl}\,\vec{F}) \cdot \vec{n}$



notice the the ellipse $T$
has the same boundary
compatible orientation
if $\vec{n}$ goes upward

so $\iint_S \mathrm{curl}\,\vec{F} \cdot \vec{n} = \int_C \vec{F} \cdot d\vec{x} = \iint_T \mathrm{curl}\,\vec{F} \cdot \vec{n}$

that could be easier to compute

⚠ It is false in general that

$$\iint_{S_1} \vec{G} \cdot \vec{n} = \iint_{S_2} \vec{G} \cdot \vec{n} \quad \text{for } G \text{ a vector field}$$

if $\partial S_1 = \partial S_2$

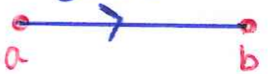
(it is true when $\vec{G} = \mathrm{curl}\,\vec{F}$) !!!!! ⚠

In this chapter, you met the following special cases of the general Stokes theorem: $\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$

## FTC:

"curve in $\mathbb{R}$"

$[a,b] \subset \mathbb{R}$




$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

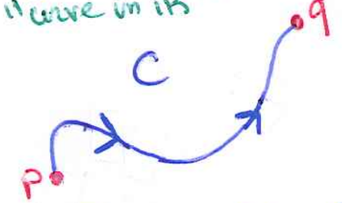
① $\int_a^b$ is the usual one-variable Riemann-Darboux integral

② $F: [a,b] \to \mathbb{R}$ $C^1$

③ $[a,b]$ is a segment line in $\mathbb{R}$

## Gradient theorem

"curve in $\mathbb{R}^m$"




$$\int_C \vec{\nabla} f \cdot d\vec{x} = f(q) - f(p)$$

① $\int_C$ is the line integral for vector fields

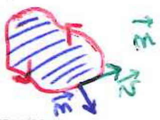
② $C$ is an oriented curve in $\mathbb{R}^m$

③ $f: \mathcal{U} \to \mathbb{R}$ $C^1$

$\mathcal{U} \subset \mathbb{R}^m$ is an open subset containing $C$

- We want the surface to be on the left

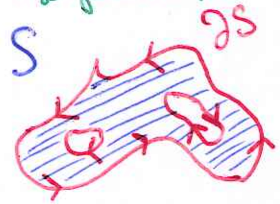
$\Leftrightarrow$ • If $\vec{v}$ is tangent compatible with the orientation, we want $\vec{n} = (v_2, -v_1)$ to point outward




$\vec{n}$ = rotation of $\vec{v}$ by $\frac{\pi}{2}$ clockwise

## Green's theorem

"surface in $\mathbb{R}^2$"




$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

or

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

① $S \subset \mathbb{R}^2$ is a planar regular region (surface in $\mathbb{R}^2$)

② $\partial S$ is piecewise smooth and positively oriented

③ $\int_{\partial S}$ is the line integral for vector fields

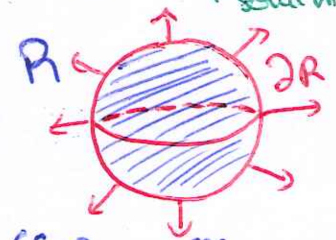
④ $\iint_S$ is the usual integral for 2-variable function $f: S \to \mathbb{R}$, $S \subset \mathbb{R}^2$

⑤ $\vec{F}: \mathcal{U} \to \mathbb{R}^2$ $C^1$

$\mathcal{U} \subset \mathbb{R}^2$ open with $S \subset \mathcal{U}$

## Divergence theorem

"solid in $\mathbb{R}^3$"




$$\iint_{\partial R} \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div}(F)$$

① $R \subset \mathbb{R}^3$ is a regular region ("solid")

② $\partial R$ is a piecewise smooth surface oriented by $\vec{n}$ the outward pointing normal unit vector.

③ $\iint_{\partial R}$ surface integral for vector fields

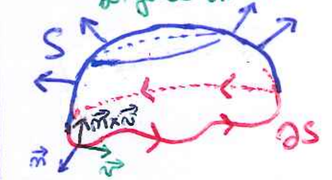
④ $\iiint_R$ usual integral for 3-variable functions $f: R \to \mathbb{R}$, $R \subset \mathbb{R}^3$

⑤ $\vec{F}: \mathcal{U} \to \mathbb{R}^3$ $C^1$

$\mathcal{U} \subset \mathbb{R}^3$ open, $R \subset \mathcal{U}$

## Stokes theorem

"surface in $\mathbb{R}^3$"




$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S (\operatorname{curl} F) \cdot \vec{n}$$

① $S \subset \mathbb{R}^3$ is an oriented surface

② $\partial S$ is the relative boundary of $S$ with the positive orientation

It is an oriented curve in $\mathbb{R}^3$

③ $\int_{\partial S}$ is the line integral for vector fields

④ $\iint_S$ is the surface integral for vector fields

⑤ $F: \mathcal{U} \to \mathbb{R}^3$ $C^1$

$\mathcal{U} \subset \mathbb{R}^3$ open, $S \subset \mathcal{U}$

We want $\vec{n} \times \vec{v}$ to point to the surface $S$