University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

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Exercise 1. Let $S \subset \mathbb{R}^n$. Prove that S is Jordan-measurable if and only if there exists a rectangle R containing S such that $\chi_S : R \to \mathbb{R}$ is integrable. *Then we define the volume of S by $v(S) := \int_S 1 := \int_R \chi_S$.*

Exercise 2. Solve the questions p50 of
http://www.math.toronto.edu/campesat/ens/1920/winter-notes.pdf

Exercise 3. Let $T \subset \mathbb{R}^n$ be a rectangle and $f : T \to \mathbb{R}$ be a non-negative integrable function.
Prove that $S = \{(x,y) \in T \times \mathbb{R} \ : \ 0 \le y \le f(x)\}$ is Jordan measurable and that $v(S) = \displaystyle\int_T f$.

Exercise 4. Compute $\displaystyle\int_S y \mathrm{d}x\mathrm{d}y$ where $S = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \le 1,\ y \ge 0\right\}$:
(1) Using Fubini's theorem. (2) Using a change of variables.

Exercise 5. Compute $\displaystyle\int_{[-1,1]\times[0,2]} \sqrt{|y-x^2|}\mathrm{d}x\mathrm{d}y$

Exercise 6. Compute $\displaystyle\int_A xy\mathrm{d}x\mathrm{d}y$ where $A \subset \mathbb{R}^2$ is the set enclosed by $x^2+y^2=1$ and $2x+y=1$ on the left.

Exercise 7.

1. Prove that $\Phi(x,y) = (x^2-y^2, 2xy)$ defines a C^1-diffeomorphism from $U = \mathbb{R}_{>0} \times \mathbb{R}$ to $\mathbb{R}^2 \setminus \{(x,0),\ x \le 0\}$. (*Hint: compute $\|\Phi(x,y)\|$ or use polar coordinates*).
2. Let $0 < a \le b$ and $S \subset \mathbb{R}^2$ be the set enclosed by $y=x$, $y^2-x^2=1$, $xy=a$ and $xy=b$.
Compute $\displaystyle\int_S (y^2-x^2)^{xy}(x^2+y^2)\mathrm{d}x\mathrm{d}y$.

Exercise 8. Compute $\displaystyle\int_0^\pi \int_0^\pi |\cos(x+y)|\mathrm{d}x\mathrm{d}y$. (*Hint: $u=x+y, v=y$*).

Exercise 9.

1. Give another proof of Leibniz rule by using Fubini's theorem:
Let $f : I\times[a,b] \to \mathbb{R}$ be continuous where I is an open interval and such that $\frac{\partial f}{\partial x}(x,y)$ exists and is continuous on $I\times[a,b]$. Then $\displaystyle\frac{\partial}{\partial x}\int_a^b f(x,y)\mathrm{d}y = \int_a^b \frac{\partial f}{\partial x}(x,y)\mathrm{d}y$.
2. Let $f : I \times J \to \mathbb{R}$ be continuous where I is an open interval and J is an interval. Assume that $\partial f/\partial x(x,y)$ exists and is continuous on $I \times J$. Let $\varphi_1, \varphi_2 : I \to J$ be differentiable. Prove that

$$\frac{\partial}{\partial x}\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\mathrm{d}y = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x,y)\mathrm{d}y + \varphi_2'(x)f(x,\varphi_2(x)) - \varphi_1'(x)f(x,\varphi_1(x))$$

Exercise 10. Define $f : [0,1]\times[0,1] \to \mathbb{R}$ by $f(x,y) = y^{-2}$ if $0<x<y<1$, $f(x,y) = -x^{-2}$ if $0<y<x<1$ and $f(x,y)=0$ otherwise.

1. Prove that $f_x(y) = f(x,y)$ and $f_y(x) = f(x,y)$ are both integrable on $[0,1]$.
2. Prove that $\int_0^1 \left(\int_0^1 f(x,y)\mathrm{d}x\right)\mathrm{d}y$ and $\int_0^1 \left(\int_0^1 f(x,y)\mathrm{d}y\right)\mathrm{d}x$ exist and are unequal.
3. Is there a contradiction with Fubini's theorem?

Exercise 11 (An integral proof of Clairaut's theorem)**.** Let $U \subset \mathbb{R}^2$ be an open subset and $f : U \to \mathbb{R}$ a C^2 function.
We want to prove Clairaut's theorem, i.e. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.
We assume by contradiction that $\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) - \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0) > 0$ for some $(x_0,y_0) \in U$.
(1) Prove that there exists a rectangle $R \subset U$ such that $\displaystyle\int_R \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\mathrm{d}x\mathrm{d}y > 0$. (2) Conclude!

Exercise 12. Define $F(x) = \displaystyle\int_0^1 \ln(x^2+y^2)\mathrm{d}y$ for $x>0$. Compute $F'(x)$.

Exercise 13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $u(x) = \int_0^x (x-y)e^{x-y} f(y)\mathrm{d}y$. Prove that $u'' - 2u' + u = f$.

Exercise 14. Review the examples at pp.85–91 from
http://www.math.toronto.edu/campesat/ens/1920/winter-notes.pdf.

Exercise 15.

1. Compute $\int_1^\infty \left(\int_1^\infty \frac{y-x}{(x+y)^3} \mathrm{d}x \right) \mathrm{d}y$ in the MAT137 (one-variable improper) sense.
2. Compute $\int_1^\infty \left(\int_1^\infty \frac{y-x}{(x+y)^3} \mathrm{d}y \right) \mathrm{d}x$ in the MAT137 (one-variable improper) sense.
3. Conclude!

Exercise 16. Prove that $\int_{\mathbb{R}^n} e^{-\alpha\|\mathbf{x}\|^2} d\mathbf{x} = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}$ where $\alpha > 0$ (improper integral).

Exercise 17. Compute the improper integral $\int_{x^2+y^2<1} \frac{\mathrm{d}x\mathrm{d}y}{(1-x^2-y^2)^\alpha}$ where $\alpha > 0$.

Exercise 18.

1. Compute the improper integral $\int_S \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x^2z^2)(1+y^2z^2)}$ where $S = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ 0 < x < 1, 0 < y < 1, z > 0\right\}$.
   (*Hint: compute* $(1+y^2z^2)x^2 - (1+x^2z^2)y^2$)
2. Deduce that $\int_0^\infty \left(\frac{\arctan t}{t}\right)^2 \mathrm{d}t = \pi \ln 2$.

Exercise 19. Compute the arclength of C where:
(1) $C = \left\{\left(e^t \cos t, e^t \sin t\right) \ : \ t \in [0, \pi/2]\right\}$ (2) $C = \left\{(\ln t, 2t, t^2) \ : \ t \in [1, e]\right\}$

Exercise 20.

1. Compute $\int_C (x+y)$ where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.
2. Compute $\int_C z$ where $C = \{(t\cos t, t\sin t, t) \ : \ t \in [0, a]\}$.

Exercise 21. Compute

1. $\int_C xe^{-y}\mathrm{d}x + \sin(\pi x)\mathrm{d}y$ where C is the portion of the parabola $y = x^2$ starting at $(0,0)$ and ending at $(1,1)$.
2. $\int_C y^2\mathrm{d}x - 2x\mathrm{d}y$ where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ oriented clockwise.
3. $\int_C y\mathrm{d}x + x\mathrm{d}y$ where C is the portion of the parabola $y = x^2$ with start point $(0,0)$ and endpoint $(2,4)$.
4. $\int_C y|y|\mathrm{d}x + x|x|\mathrm{d}y$ where C is the boundary of $\left\{(x,y) \in \mathbb{R}^2 \ : \ |x| + |y| \leq 1\right\}$ positively oriented.

Exercise 22. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y) = (xy^2 + 3x^2y, x^3 + yx^2)$.

1. Prove that there exists $f : \mathbb{R}^2 \to \mathbb{R}$ C^1 such that $F = \nabla f$.
2. Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ where C is the given by the portion of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$ then the line segment from $(2,4)$ to $(4,4)$ and finally the line segment from $(4,4)$ to $(7,8)$.

Exercise 23. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field and $C \subset \mathbb{R}^n$ be a piecewise smooth curve. Prove that

$$\left| \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \right| \leq \left(\max_{x \in C} \|\mathbf{F}(x)\| \right) \mathscr{L}(C) \qquad (\textit{where } \mathscr{L}(C) \textit{ is the arclength of } C)$$

Exercise 24.

1. Compute $\iint_S z^2$ for $S = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\right\}$
2. Compute $\iint_S \sqrt{x^2 + y^2 + 1}$ for $S = \{(r\cos\theta, r\sin\theta, \theta) \ : \ r \in [0,1], \ \theta \in [0, 2\pi]\}$ (*helicoid, sketch it!*)

Exercise 25. Assume that $C = \{(0, y(t), z(t)) \ : \ t \in [a,b]\}$ is a curve in the yz-place where $y(t) \geq 0$. We denote by S the surface of revolution obtained by revolving C around the z-axis.

1. Find a parametrization of S.

2. Find an expression for $\iint_S f$ where $f : S \to \mathbb{R}$ is continuous.
3. Find the area of the cylinder of radius $r > 0$ and height $h > 0$.
4. Find the area of the cone of base radius $r > 0$ and height $h > 0$.
5. Find the area of the torus obtained by revolving the circle situated in the xy-plan whose center is on the positive part of the y-axis at distance b to the origin and of radius a where $0 < a < b$.

Exercise 26. Compute $\iint_S \mathbf{F} \cdot \mathbf{n}$ where

1. $\mathbf{F}(x, y, z) = (x, y^2, z)$ and S is the triangle whose vertices are obtained by intersecting the plane $x + y + z = 1$ with the axes and whose orientation is obtained by taking the normal vector pointing *away from the origin*.
2. $\mathbf{F}(x, y, z) = (x^2, z, -y)$ and S is the unit sphere whose orientation is obtained by the outward pointing normal unit vector.
3. $\mathbf{F}(x, y, z) = (x, y, z^2)$ where S is the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 1$ and including the top and the bottom always with the outward normal unit vector orientation.

Exercise 27.

1. Compute $\int_C \frac{-y}{x^2+4y^2}\mathrm{d}x + \frac{x}{x^2+4y^2}\mathrm{d}y$ where C is the unit circle with the counter-clockwise orientation.
2. Prove that $\frac{\partial}{\partial x}\left(\frac{x}{x^2+4y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+4y^2}\right) = 0$.
3. Is there any contradiction with Green's theorem?

Exercise 28.

1. Compute $\int_C \left(1 - x^2\right) y\mathrm{d}x + \left(1 + y^2\right) x\mathrm{d}y$ where C is the circle centered at $\mathbf{0}$ of radius $a > 0$ with the clockwise orientation.
2. Compute $\int_C \left(-x^2 y\right) \mathrm{d}x + \left(xy^2\right) \mathrm{d}y$ where C is the boundary of the annulus centered at $\mathbf{0}$ of radii 1 and 2 positively oriented (draw C with its orientation first!).

Exercise 29.

1. Compute the area between the x-axis and one arch of the cycloid $\sigma(\theta) = (R(\theta - \sin\theta), R(1 - \cos\theta))$.
2. Compute the area enclosed within the cardioid $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$ where $a > 0$.
(*Hint: find a polar equation $r(\theta) = \cdots$ in order to draw the cardioid and find a parametrization*).

Exercise 30.
Prove that $\int_C \left(yx^3 + xe^y\right) \mathrm{d}x + \left(xy^3 + ye^x - 2y\right) \mathrm{d}y = 0$ where C is any closed curve symmetric w.r.t. the origin.

Exercise 31. Let $\mathbf{F}(x, y, z) = (x^2, xyz, yz^2)$. Compute $\operatorname{div}\mathbf{F}$, $\operatorname{curl}\mathbf{F}$ and $\Delta\mathbf{F}$.

Exercise 32. Let $f, g : U \to \mathbb{R}$ and $\mathbf{F}, \mathbf{G} : U \to \mathbb{R}^n$ be C^1 where $U \subset \mathbb{R}^n$ is open. Prove the following product rules.
(1) $\nabla(fg) = f\nabla g + g\nabla f$
(2) $\operatorname{div}(f\mathbf{G}) = f \operatorname{div}\mathbf{G} + (\nabla f) \cdot \mathbf{G}$
(3) $\operatorname{curl}(f\mathbf{G}) = f \operatorname{curl}\mathbf{G} + (\nabla f) \times \mathbf{G}$ $(n = 3)$
(4) $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\operatorname{curl}\mathbf{F}) - \mathbf{F} \cdot (\operatorname{curl}\mathbf{G})$ $(n = 3)$
Can you see with no computation why the formula in (4) couldn't be true with a "+"?

Exercise 33. Find all the C^2 vector fields $\mathbf{F} : U \to \mathbb{R}^3$ ($\emptyset \neq U \subset \mathbb{R}^3$ open) such that $\operatorname{curl}\mathbf{F}(x, y, z) = (x, y, z)$.
(*Hint: think before trying to solve!*)

Exercise 34. Let $\mathbf{F} : U \to \mathbb{R}^3$ be a C^2 vector field ($U \subset \mathbb{R}^3$ open). Prove that $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla(\operatorname{div}\mathbf{F}) - \Delta\mathbf{F}$.
The vector Laplacian is usually used to lighten the writing of some physics equations by simplifying these double curls: for instance you derive the electromagnetic wave equations by applying curl to the two Maxwell equations involving a curl.

Exercise 35. Set $\mathbf{r}(x, y, z) = (x, y, z)$ and let $\mathbf{a} \in \mathbb{R}^3$. (1) Compute $\operatorname{div}\mathbf{r}$, $\operatorname{curl}\mathbf{r}$, and $\nabla(\|\mathbf{r}\|^2)$.
(2) Prove the identities: $\operatorname{curl}(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$, $\operatorname{div}((\mathbf{a} \cdot \mathbf{r})\mathbf{a}) = \|\mathbf{a}\|^2$, $\operatorname{div}((\mathbf{a} \times \mathbf{r}) \times \mathbf{a}) = 2\|\mathbf{a}\|^2$, $\operatorname{div}(\|\mathbf{r}\|^n(\mathbf{a} \times \mathbf{r})) = 0$

Exercise 36. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = (xy^2 + 3x^2y, x^3 + yx^2)$. For (3) use all the C from Exercise 21.
(1) Quickly prove that there exists $f : \mathbb{R}^2 \to \mathbb{R}$ C^1 such that $\mathbf{F} = \nabla f$. (2) Find a suitable f. (3) Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$.

Exercise 37. According to Ex27(1), the vector field $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}\right)$ is not conservative. But according to Ex27(2) it seems to satisfy the assumption of Poincaré Lemma. Why isn't there any contradiction?