The divergence theorem

(also called Gauss theorem or Green - Ostrogradski theorem)

[Not part of HAT237: it is again a consequence of the general
Stokes theorem $\int_\Omega d\omega = \int_{\partial\Omega} \omega$

Theorem: $R \subset \mathbb{R}^3$ a regular region with piecewise smooth boundary
(ie ∂R is as in the surface integral section)
We assume that ∂R is oriented by the normal outward pointing unit vector $\vec{n}(x)$

$$F : \mathcal{U} \longrightarrow \mathbb{R}^3 \quad C^1, \ \mathcal{U} \subset \mathbb{R}^3 \text{ open}, \ R \subset \mathcal{U}$$

Then
$$\underbrace{\iint_{\partial R} \vec{F} \cdot \vec{n}}_{\text{Surface integral for vector fields}} = \underbrace{\iiint_R \operatorname{div} F}_{\text{usual integral for 3 variables and } \operatorname{div} F : R \to \mathbb{R}}$$

↳ usual integral for 3 variables and $\operatorname{div} F : R \longrightarrow \mathbb{R}$

Surface integral for vector fields

Δ We are not proving this theorem

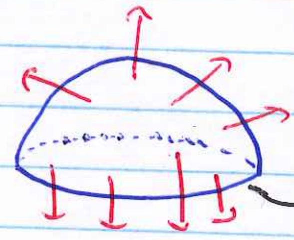
① If each part of the boundary is elementary
ie $\{\varphi_1(x,y) \leq z \leq \varphi_2(x,y)\} = \{\psi_1(y,z) \leq x \leq \psi_2(y,z)\}$
$= \{\tau_1(x,z) \leq y \leq \tau_1(x,z)\}$
then we may follow the proof of Green theorem

② The general case is too difficult for MAT237

$\square$

Ex: Let $R = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 2,\ z \geq 0\}$

and $S = \partial R$ oriented by taking the unit outward pointing normal vector $\vec{n}$

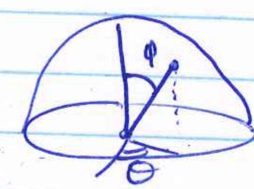


on this circle the normal vector is not well defined but it's a curve so it has an integral of 0 (zero content)

$F(x,y,z) = ((2x+y)z, 0, z)$

Compute $\iint_S \vec{F} \cdot \vec{n}$.

$$\iint_S \vec{F}\cdot\vec{n} = \iiint_R \operatorname{div} F \overset{\text{divergence theorem}}{=} \iiint_R 2z+1$$



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (2r\cos\varphi + 1)\, r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/2}\int_0^{\sqrt{2}} r^3 \sin(2\varphi) + r^2 \sin\varphi \, dr\, d\varphi$$

$$= 2\pi \int_0^{\sqrt{2}} r^3 + r^2 \, dr$$

$$= 2\pi \left(1 + \frac{2\sqrt{2}}{3}\right)$$

The divergence theorem allows to give a physical interpretation of the divergence operator.

$\mathcal{U} \subset \mathbb{R}^3$ open, $F : \mathcal{U} \longrightarrow \mathbb{R}^3$ C^1, $p \in \mathcal{U}$

then if $r > 0$ is small $\operatorname{div} F(x) \simeq \operatorname{div} F(p)$ for $x \in \bar{B}(p,r)$
(continuity)

so that:

$$\iiint_{\bar{B}(p,r)} \operatorname{div} F \simeq \operatorname{div} F(p) \cdot \iiint_{\bar{B}(p,r)} 1 = \frac{4}{3}\pi r^3 \operatorname{div} F(p)$$

ie $$\operatorname{div} F(p) \simeq \frac{3}{4\pi r^3} \iiint_{\bar{B}(p,r)} \operatorname{div} F$$

$$= \frac{3}{4\pi r^3} \iint_{\partial \bar{B}(p,r)} \vec{F} \cdot \vec{n}$$

By the way this is independant of the coordinates so $\operatorname{div} F$ is the same in any coordinate systems

Ccl: $$\operatorname{div} F(p) = \lim_{r \to 0} \frac{3}{4\pi r^3} \iint_{\partial \bar{B}(p,r)} \vec{F} \cdot \vec{n}$$

ie $\operatorname{div} F(p)$ measures the outward flux near p per unit of volume per unit of time

so if $\operatorname{div} F(p) > 0$, p is a source, the flux is outgoing 

if $\operatorname{div} F(p) < 0$, p is a sink, the flux is ingoing 

Gauss' law: $R \subset \mathbb{R}^3$ a regular region with piecewise smooth boundary s.t. $\vec{0} \notin \partial R$
We assume that ∂R is oriented by the normal outward pointing unit vector field.

Then $$\iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} \cdot \vec{n} = \begin{cases} 4\pi & \text{if } \vec{0} \in R \\ 0 & \text{otherwise} \end{cases}$$

where $\vec{r}(x,y,z) = (x,y,z)$

Δ First case: $\vec{0} \notin R$

$$\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) = \operatorname{div}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \|\vec{r}\|^2 - 3x^2 + \|\vec{r}\|^2 - 3y^2 + \|\vec{r}\|^2 - 3z^2$$
$$= 3\|\vec{r}\|^2 - 3\|\vec{r}\|^2$$
$$= 0$$

and it is true on R because $0 \notin R$

By the divergence theorem: $\iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} = \iiint_R 0 = 0$

Second case: $\vec{0} \in R$

We can no longer directly apply the divergence theorem since $\frac{\vec{r}}{\|\vec{r}\|}$ is not defined at $\vec{0} \in R$

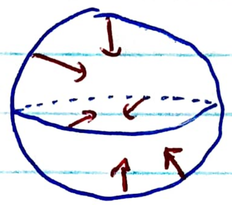
Since $\vec{0} \in R \setminus \partial R$, we know that $\vec{0} \in \mathring{R}$
hence $\exists \varepsilon > 0$ s.t.

$$B(\vec{0}, \varepsilon) \subset \mathring{R} \subset \overline{\mathring{R}} = R$$

$\hookrightarrow \mathring{R}$ open $\hookrightarrow$ always true $\rightarrow R$ is a regular region

We define $R' = R \setminus B(0,\varepsilon)$

then $\partial R' = \partial R \cup \partial B(0,\varepsilon)$ with the orientation pointing outward from R' so

$\partial B(0,\varepsilon)$:  (inward for the ball) ⚠

By the divergence theorem:

$$0 = \iiint_{R'} 0 = \iiint_{R'} \mathrm{div}\left(\frac{\vec r}{\|\vec r\|^3}\right) = \iint_{\partial R} \frac{\vec r}{\|\vec r\|^3} - \iint_{\partial B(0,\varepsilon)} \frac{\vec r}{\|\vec r\|^3}$$

↳ outward pointing for the ball here

$$\Rightarrow \iint_{\partial R} \frac{\vec r}{\|\vec r\|^3} = \iint_{\partial B(0,\varepsilon)} \frac{\vec r}{\|\vec r\|^3}$$

$\partial B(0,\varepsilon) = \{(\varepsilon\cos\theta\sin\varphi, \varepsilon\sin\theta\sin\varphi, \varepsilon\cos\varphi)\}$
$\theta \in [0, 2\pi]$, $\varphi \in [0,\pi]$

$\partial_\theta \sigma \times \partial_\varphi \sigma = -\sin\varphi\, \varepsilon\, \vec r(\sigma(\theta,\varphi))$

$$= \int_0^\pi \int_0^{2\pi} \frac{\vec r(\sigma(\theta,\varphi))}{\|\vec r(\sigma(\theta,\varphi))\|^3} \cdot (\sin\varphi\, \varepsilon\, \vec r(\sigma(\theta,\varphi)))$$

< 0 inward

but we want outward so $\times -1$

$$= \int_0^\pi \int_0^{2\pi} \sin\varphi \frac{\cancel{\|\vec r\|^2}\, \varepsilon}{\|\vec r\|^3}\, d\theta\, d\varphi$$

$\|\vec r(\sigma(\theta,\varphi))\| = \varepsilon$

$$= \int_0^\pi \int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi = 2\pi \left[-\cos\varphi\right]_0^\pi = 4\pi$$

□

$$\iint_{\partial B(0,1)} \frac{\vec{r}}{\|\vec{r}\|^3} = 4\pi \neq 0$$

whereas $\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) = 0$

because we can't apply the divergence theorem to $B(\vec{0},1)$ since $\vec{0} \in B(\vec{0},1)$ but $\frac{\vec{r}}{\|\vec{r}\|^3}$ is not defined at $\vec{0}$

$\rightarrow$ Be careful before applying the divergence theorem

<u>Example</u>: We can use the divergence theorem to compute a volume using a surface integral (in the same way that we use Green's thm to compute an area using a line integral)

Let $R \subset \mathbb{R}^3$ be a regular region with a piecewise smooth boundary.
We assume that ∂R is oriented using the <u>outward</u> normal unit vector $\vec{n}$.
Let $\vec{r}(x,y,z) = (x,y,z)$

$$\mathcal{V}(R) = \iiint_R 1 = \iiint_R \frac{\operatorname{div}(\vec{r})}{3}$$

$$= \frac{1}{3}\iiint_R \operatorname{div}(\vec{r})$$

$$= \frac{1}{3}\iint_{\partial R} \vec{r}\cdot\vec{n} \quad \text{by the divergence theorem}$$

↳ outward orientation

$$\boxed{\mathcal{V}(R) = \frac{1}{3}\iint_{\partial R} \vec{r}\cdot\vec{n}}$$