Gradient, curl, divergence

Def: $\mathcal{U} \subset \mathbb{R}^n$ open, $f : \mathcal{U} \longrightarrow \mathbb{R}$ C^1 → real valued

The gradient of f at $x \in \mathcal{U}$ is

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right) \in \mathbb{R}^n$$

so that $\nabla f : \mathcal{U} \longrightarrow \mathbb{R}^n$ C^0

→ domain and codomain have same dimensions

Def: $\mathcal{U} \subset \mathbb{R}^n$ open, $F : \mathcal{U} \longrightarrow \mathbb{R}^n$ C^1

The divergence of F at $x \in \mathcal{U}$ is

$$\operatorname{div} F(x) := \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) + \dots + \frac{\partial F_n}{\partial x_n}(x) \in \mathbb{R}$$

so that $\operatorname{div} F : \mathcal{U} \longrightarrow \mathbb{R}$ C^0

Notation / Mnemonic device:

if you write $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $F = (F_1, \dots, F_n)$

then $\operatorname{div} F = \nabla \cdot F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i)$

↳ dot product

So it is common to denote $\operatorname{div}(F) = \nabla \cdot F$

Be careful, that's just an abuse of notation and a good mnemonic device: $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is not really a vector and $\partial/\partial x_i$ is not a scalar only for $n=3$ codomain is also $\mathbb{R}^3$

Def: $\mathcal{U} \subset \mathbb{R}^3$ open, $F: \mathcal{U} \longrightarrow \mathbb{R}^3$ C^1

The curl of F at $x \in \mathcal{U}$ is

$$\operatorname{curl} F(x) := \begin{pmatrix} \frac{\partial F_3}{\partial x_2}(x) - \frac{\partial F_2}{\partial x_3}(x) \\ \frac{\partial F_1}{\partial x_3}(x) - \frac{\partial F_3}{\partial x_1}(x) \\ \frac{\partial F_2}{\partial x_1}(x) - \frac{\partial F_1}{\partial x_2}(x) \end{pmatrix} \in \mathbb{R}^3$$

So that $\operatorname{curl} F: \mathcal{U} \longrightarrow \mathbb{R}^3$ C^0

Notation / Mnemonic device:

if you write $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $F = (F_1, F_2, F_3)$

then $\operatorname{curl} F = \underbrace{\nabla \times F}_{\text{cross product}}$

So it is common to write $\operatorname{curl} F = \nabla \times F$

(Again, it is just an abuse of notation and a very good mnemonic device)

Def: $\mathcal{U} \subset \mathbb{R}^n$ open, $f: \mathcal{U} \to \mathbb{R}$ C^2 C^2!

We define the Laplacian or Laplace operator of f at x by

$$\Delta f(x) := \frac{\partial^2 f}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 f}{\partial x_n^2}(x) \in \mathbb{R}$$

so that $\Delta f: \mathcal{U} \to \mathbb{R}$ C^0

Comment: $\Delta f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f)$

so it is common to use the notation $\nabla^2 f = \Delta f$

Def. $\mathcal{U} \subset \mathbb{R}^3$ open, $F : \mathcal{U} \longrightarrow \mathbb{R}^3$ C^2

The vector Laplacian of F at $x \in \mathcal{U}$ is

$$\Delta F(x) = (\Delta F_1(x), \Delta F_2(x), \Delta F_3(x)) \in \mathbb{R}^3$$

where $F = (F_1, F_2, F_3)$

Comment: $\Delta F = \operatorname{grad}(\operatorname{div} F) - \operatorname{curl}(\operatorname{curl} F) = \nabla(\nabla \cdot F) - \nabla \times (\nabla \times F)$

Product rules: $f, g : \mathbb{R}^n \to \mathbb{R}$, $F, G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ C^1

($n = 3$ when a curl is involved)

$$\left.\begin{aligned}
\nabla(fg) &= f \nabla g + g \nabla f \\
\operatorname{div}(fG) &= f \operatorname{div}(G) + (\nabla f) \cdot G \\
\operatorname{curl}(fG) &= f \operatorname{curl} G + (\nabla f) \times G \\
\operatorname{div}(F \times G) &= G \cdot (\operatorname{curl} F) - F \cdot (\operatorname{curl} G)
\end{aligned}\right\} \text{essentials}$$

$$\text{a little bit less} \left\{\begin{aligned}
\operatorname{curl}(F \times G) &= (G \cdot \nabla) F + (\operatorname{div} G) F - (F \cdot \nabla) G - (\operatorname{div} F) G \\
\nabla(F \cdot G) &= (G \cdot \nabla) F + G \times (\operatorname{curl} F) + (F \cdot \nabla) G + F \times (\operatorname{curl} G)
\end{aligned}\right.$$

$$(F \cdot \nabla) G = \left(\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \right) G = \left(F_1 \frac{\partial}{\partial x_1}, F_2 \frac{\partial}{\partial x_2}, F_3 \partial/\partial x_3 \right) G$$

(Mnemonic device)

$$= F_1 \frac{\partial G}{\partial x_1} + F_2 \frac{\partial G}{\partial x_2} + F_3 \frac{\partial G}{\partial x_3} \in \mathbb{R}^3$$

$$\partial G/\partial x_i = \left(\frac{\partial G_1}{\partial x_i}, \frac{\partial G_2}{\partial x_i}, \frac{\partial G_3}{\partial x_i} \right) \in \mathbb{R}^3$$

Assume that $n = 3$ then:

For $f : \mathbb{R}^3 \to \mathbb{R}$ C^2, $\operatorname{curl}(\nabla f) = \vec{0}$

For $F : \mathbb{R}^3 \to \mathbb{R}^3$ C^2, $\operatorname{div}(\operatorname{curl} F) = 0$

That you may summarize by noticing that two successive arrows in the following diagram give 0:

$$\left\{ \begin{array}{c} \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{array} \right\} \xrightarrow{\nabla} \left\{ \begin{array}{c} \text{vector fields} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{array} \right\} \xrightarrow{\text{curl}} \left\{ \begin{array}{c} \text{vector fields} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{array} \right\} \xrightarrow{\text{div}} \left\{ \begin{array}{c} \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{array} \right\}$$

<u>Not part of MAT 237</u>: When you will learn differential forms and de Rham cohomology, come back here and compare the above diagram with:

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

where $\Omega^k(\mathbb{R}^3) = \{ k\text{-differential forms on } \mathbb{R}^3 \}$

and d is the exterior derivative and $d \circ d = 0$
(exact forms are closed)

<u>Spoiler</u>: ∇, curl, div correspond to the exterior derivative using the following bases:

$$\Omega^1(\mathbb{R}^3) = \langle dx, dy, dz \rangle$$
$$\Omega^2(\mathbb{R}^3) = \langle dydz, dzdx, dxdy \rangle$$

<u>HOMEWORK</u>: Questions from 5.4