Surface integrals

Convention: in this section, by a surface I mean:

$$S = \{\sigma(t) : t \in T\} \subset \mathbb{R}^3$$

where $\sigma : \mathcal{U} \to \mathbb{R}^3$ is C^1, $\mathcal{U} \subset \mathbb{R}^2$ is open, $T \subset \mathcal{U}$ is Jordan measurable, σ is injective on T, and $(\partial_1 \sigma, \partial_2 \sigma)$ are linearly independent except on a set having zero content

Notation: $\partial_1 \sigma = \left(\frac{\partial \sigma_1}{\partial x}, \frac{\partial \sigma_2}{\partial x}, \frac{\partial \sigma_3}{\partial x}\right)$, $\partial_2 \sigma = \left(\frac{\partial \sigma_1}{\partial y}, \frac{\partial \sigma_2}{\partial y}, \frac{\partial \sigma_3}{\partial y}\right)$

Def: Let $S \subset \mathbb{R}^3$ be as above, $f : S \to \mathbb{R}$ C^0
We define the surface integral of f over S by

$$\iint_S f = \iint_T f(\sigma(u,v)) \, \| \partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) \| \, du \, dv$$

Comment 1: It doesn't depend on the parametrization of S

Comment 2: In general S may not admit a "global" parametrization
It is possible to give a more general definition
but it outreaches MAT237

The most general case we will consider is

$S = \bigcup_{i=1}^{N} S_i$ where

(1) S_i is as above

(2) $S_i \cap S_j = \emptyset$ or $S_i \cap S_j$ is a curve

then $\int_S f = \sum_{i=1}^{N} \int_{S_i} f$ (By (2) the curves counted twice have a zero integral)

Def: $S \subset \mathbb{R}^3$ be a surface as before

The area of S is $A(S) := \iint_S 1 = \iint_T \|\partial_1 \sigma \times \partial_2 \sigma\|$

Ex: $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

$\sigma(\varphi, \theta) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi) \qquad \theta \in [0, 2\pi], \varphi \in [0, \pi]$

$$\|\partial_1 \sigma \times \partial_2 \sigma\| = \left\| \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ -\sin\varphi \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} -\cos\theta \sin^2\varphi \\ -\sin\theta \sin^2\varphi \\ -\sin\varphi \cos\varphi \end{pmatrix} \right\|$$

$$= \left(\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2}$$

$$= \left(\sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2} = \sin\varphi$$

So $A(S) = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = 4\pi$

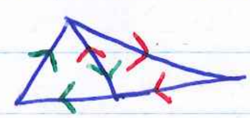
Orientation of a surface in $\mathbb{R}^3$

There are several equivalent ways to define the orientability of a surface:
$S \subset \mathbb{R}^3$ is orientable if

1. There exists a continuous normal vector field:
$$\vec{n} : S \to \mathbb{R}^3 \quad C^0$$
s.t. $\forall p \in S$, $\vec{n}(p) \neq \vec{0}$ and $\vec{n}(p)$ is orthogonal to S at p.

2. We may cover S by local parametrizations whose orientations agree on their intersections

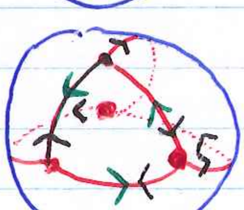
3. We may decompose S into triangles with compatible orientation along a common edge:



we want both triangle to give an opposite orientation on the common edge

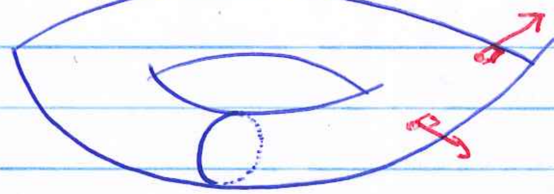
Remark: It's a subtle notion, so I prefer to keep it informal to avoid technical details

Eg. Sphere: $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ is orientable
take $\vec{n}(p)$ = outward pointing unit normal vector at p



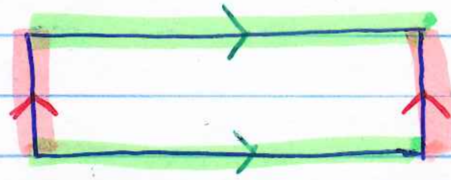
a triangulation with 4 triangles

Eg: Torus : T =

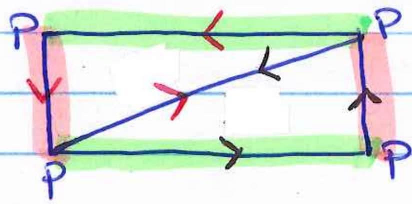


$\vec{n}(p)$ = unit normal vector at p which is pointing outward

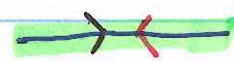
T may be obtained by gluing the edges of a band of paper like that:



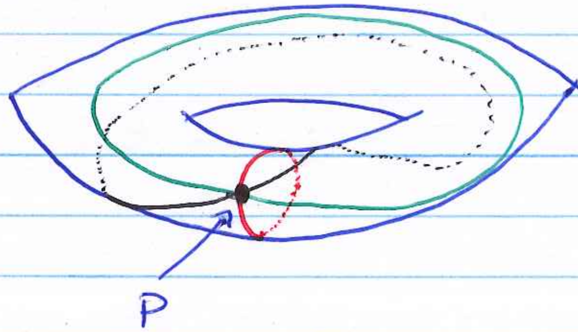
It is easier to see a triangulation this way:



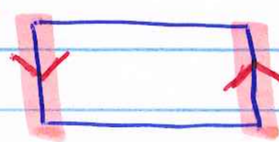
(notice that the opposite orientation condition is satisfied by the edges glued together)



and



Eg: some surfaces are not orientable, for instance the Möbius band that you can construct yourself:



Comment: If a surface is orientable, it admits only two orientations in each connected component.

So # of orientation = $2^{\#\text{connected components}}$

Def: Let $S \subset \mathbb{R}^3$ be an oriented surface whose orientation is given by $\vec{n}: S \to \mathbb{R}^3$ a continuous unit ($\|\vec{n}\| = 1$) normal vector field
Let $F: S \to \mathbb{R}^3$ be a C^0 vector field

The surface integral of F along S oriented by $\vec{n}$ is

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{n}$$

(Here $x \mapsto \vec{F}(x) \cdot \vec{n}(x)$ is a real valued function, so the surface integral of $\vec{F} \cdot \vec{n}$ is well defined)

Comment: The surface integral of a vector field is not defined along non-orientable surfaces.

Def: Let $S \subset \mathbb{R}^3$ be a surface as above together with an orientation given by a continuous normal vector field $\vec{n}: S \to \mathbb{R}^3$

Notice that $\partial_1 \sigma \times \partial_2 \sigma$ is normal to S, so either

① $\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) = \lambda \vec{n}(\sigma(u,v))$, $\lambda > 0$

ie the parametrization is compatible with the orientation

or ② $\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) = \lambda \vec{n}(\sigma(u,v))$, $\lambda < 0$

ie the parametrization gives the opposite orientation

Proposition: If $S \subset \mathbb{R}^3$ is an oriented surface together with a parametrization $\sigma: T \to S$ compatible with its orientation, then

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{n} = \iint_T F(\sigma(u,v)) \cdot (\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v))\, du\, dv$$

Δ Indeed, then $\vec{n}(\sigma(u,v)) = \dfrac{\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)}{\|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)\|}$, thus

$$\iint_S \vec{F}\cdot\vec{n} = \iint_T \big(F(\sigma(u,v))\cdot n(\sigma(u,v))\big)\,\|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)\|\,du\,dv$$

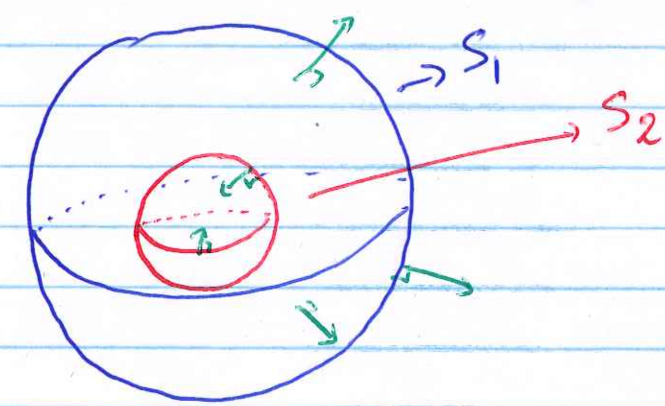
$$= \iint_T F(\sigma(u,v))\cdot\big(\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)\big)\,du\,dv$$

$\square$

Comment: If a surface is part of the boundary of a regular region in $\mathbb{R}^3$ then it is always orientable.
The usual orientation consists in taking pointwise the normal vector pointing outward.

Ex: $R = \{(x,y,z) : 1 \leq x^2+y^2+z^2 \leq 4\}$

$\partial R = S_1 \cup S_2$



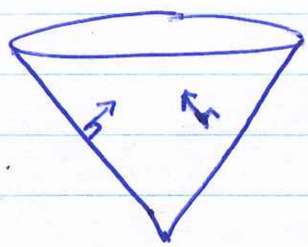
Example : • $S = \{ x^2 + y^2 = z^2,\ 0 \leq z \leq 1 \}$

• orientation given by $\vec{n}$ pointing to the z-axis

• $F(x,y,z) = (xz, yz, y)$

Compute $\iint_S \vec{F} \cdot \vec{n}$

Δ



(we could also have used $\sigma(x,y) = (x, y, \sqrt{x^2+y^2})$ for $x^2+y^2 \leq 1$)

$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $r \in [0,1]$, $\theta \in [-\pi, \pi]$

$$\partial_1 \sigma = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \qquad \partial_2 \sigma = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

$$\partial_1 \sigma \times \partial_2 \sigma = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$$

Gives the good orientation. (check it)

$$\iint_S \vec{F} \cdot \vec{n} = \int_{-\pi}^{\pi} \int_0^1 \begin{pmatrix} r^2\cos\theta \\ r^2\sin\theta \\ r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix} dr\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_0^1 \underbrace{-r^3\cos^2\theta - r^3\sin^2}_{-r^3} + r^2\sin\theta \, dr\, d\theta$$

$$= \int_{-\pi}^{\pi} -\frac{1}{4} + \frac{1}{3}\sin\theta \, d\theta$$

$$= -\frac{\pi}{2}$$

Addendum 1 : Why is there a cross product in the surface integral of a real-valued function ?

Answer 1 : from a mathematics point of view :

For the same reason we have a Jacobian determinant in the change of variables formula :

$$\iint_S f = \iint_T f(\sigma(u,v)) \underbrace{\|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)\|}\, du\, dv$$

this factor ensures that the value doesn't depend on the "speed" of the parametrization (and hence on the choice of the parametrization)

ⓐ You can repeat the heuristic idea I gave at the beginning of the CV (p58 of the notes) and see that it is the "good" factor to add.

ⓑ You can compute directly :

$\sigma_1 : T_1 \longrightarrow S$, $\sigma_2 : T_2 \longrightarrow S$, parametrizations

$\varphi : T_2 \longrightarrow T_1$ C^1-diffeomorphism (actually $\varphi : \mathcal{U}_2 \to \mathcal{U}_1$ open s.t. $\varphi(T_2) = T_1$ ...)

s.t. $\sigma_2 = \sigma_1 \circ \varphi$

$$\iint_{T_2} f\circ\sigma_2 \,\|\partial_1\sigma_2 \times \partial_2\sigma_2\| = \iint_{T_2} f(\sigma_1(\varphi)) \,\|\partial_1\sigma_1(\varphi) \times \partial_2\sigma_1(\varphi)\|\, |\det D\varphi|$$

Compute $\partial_1(\sigma_1\circ\varphi)$

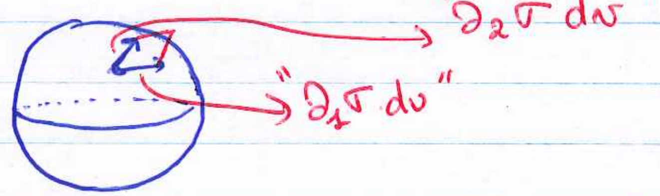
$\partial_2(\sigma_1\circ\varphi)$

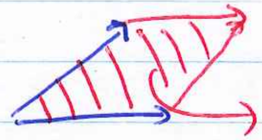
and simplify ...

$$\boxed{CV} \longrightarrow = \iint_{T_1} f\circ\sigma_1 \,\|\partial_1\sigma_1 \times \partial_2\sigma_1\|$$

Answer 2: from a "physics" point of view.

We use the parametrization to "locally flatten" S by approximating it with parallelograms:





the area here is $\|(\partial_1\sigma_1 du)\times(\partial_2\sigma_2 dv)\|$

So we get some kind of Riemann sum

$$\sum_R f(p)\, \partial(R)$$

$p \in R$

↳ R is the parallelogram

and the limit when $\partial(R) \longrightarrow 0$

gives the integral

x——x

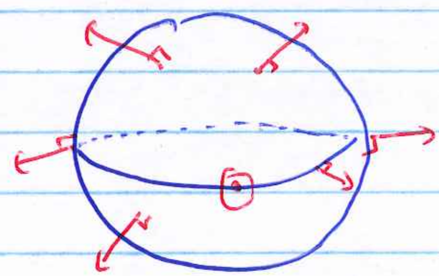
Not part of MAT 237: come back here when you'll learn about "the first differential form" and "Gauss Theorema Egregium" in Riemannian geometry:

$$\|\partial_1\sigma \times \partial_2\sigma\| = \sqrt{(\partial_1\sigma\cdot\partial_1\sigma)(\partial_2\sigma\cdot\partial_2\sigma) - (\partial_1\sigma\cdot\partial_2\sigma)^2}$$
$$= \sqrt{EG - F^2}$$

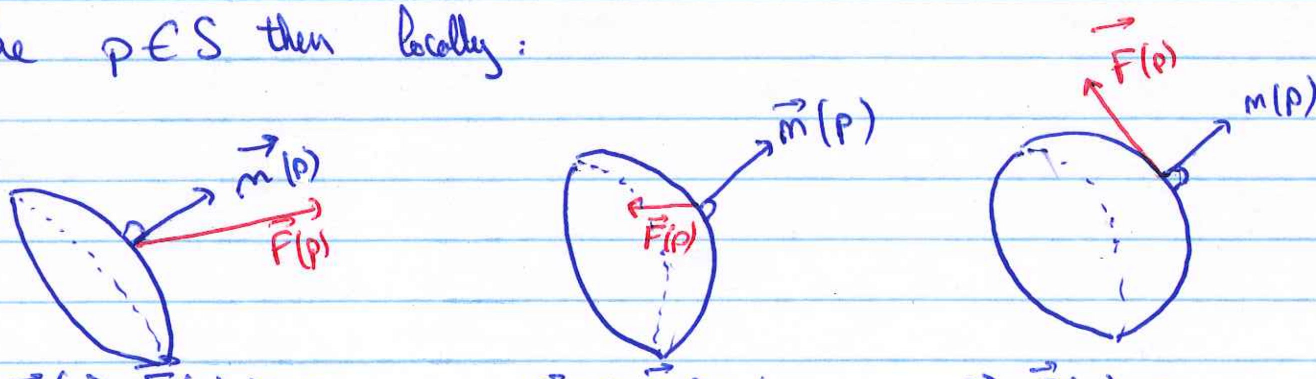
Addendum 2 : What's the physics interpretation of the surface integral of vector field?

Assume that we have a fluid in motion in the space and denote by $\vec{F}(p)$ the velocity of the fluid at p

We have S an oriented surface, let's say a sphere with orientation given by outward pointing normal unit vector



Take $p \in S$ then locally:



$\vec{n}(p) \cdot \vec{F}(p) > 0$ "the fluid goes out" at p

$\vec{n}(p) \cdot \vec{F}(p) < 0$ "the fluid goes in" at p

$\vec{n} \cdot \vec{F}(p) = 0$ "the fluid doesn't cross S at p"