University of Toronto – MAT137Y1 – LEC0501

Calculus!

Improper integrals in one variable

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1 Definitions of improper integrals

Definition 1.1. Let $f : [a, +\infty) \to \mathbb{R}$ be a function which is integrable on $[a, c]$ for any $c > a$, then we set

$$\int_a^{+\infty} f(x)dx := \lim_{c \to +\infty} \int_a^c f(x)dx$$

whenever it makes sense.

Definition 1.2. Let $f : (-\infty, b] \to \mathbb{R}$ be a function which is integrable on $[c, b]$ for any $c < b$, then we set

$$\int_{-\infty}^b f(x)dx := \lim_{c \to -\infty} \int_c^b f(x)dx$$

whenever it makes sense.

Definition 1.3. Let $f : [a, b) \to \mathbb{R}$ be a function which is integrable on any $[a, c]$ for $c \in (a, b)$, then we set

$$\int_a^b f(x)dx := \lim_{c \to b^-} \int_a^c f(x)dx$$

whenever it makes sense.

Definition 1.4. Let $f : (a, b] \to \mathbb{R}$ be a function which is integrable on any $[c, b]$ for $c \in (a, b)$, then we set

$$\int_a^b f(x)dx := \lim_{c \to a^+} \int_c^b f(x)dx$$

whenever it makes sense.

Definition 1.5. Let $f : (a, b) \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, d] \subset (a, b)$ where $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b = +\infty$.

We say that $\int_a^b f(x)dx$ is convergent if there exists $c \in (a, b)$ such that the improper integrals $\int_a^c f(x)dx$ of $f : (a, c] \to \mathbb{R}$ and $\int_c^b f(x)dx$ of $f : [c, b) \to \mathbb{R}$ are both convergent and then we set

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Remark 1.6. In the above definition, you need to study the two bounds **separately!**

Note that if $\int_a^b f(x)dx$ is convergent for some $c \in (a, b)$ then it is for any $c \in (a, b)$ and its value doesn't depend on the choice of c.

Example 1.7. The integral $\displaystyle\int_{-\infty}^{+\infty} xdx$ is not convergent although $\displaystyle\lim_{c \to +\infty} \int_{-c}^c xdx = 0$.

2 The MCT for functions

The following result relies on the Dedekind-completeness of $\mathbb{R}$.

Theorem 2.1 (The MCT – Part 1: the bounded case)**.**
Let $F : [a,b) \to \mathbb{R}$ be a function where either $b \in \mathbb{R}_{>a}$ or $b = +\infty$.
If F is non-decreasing and bounded from above then $\lim\limits_{x\to b^-} F(x)$ exists and moreover $\lim\limits_{x\to b^-} F(x) = \sup\limits_{[a,b)} F$.

Proof. The set $S = \{F(x),\ x \in [a,b)\}$ is not empty since it contains $F(a)$, and, is bounded from above since F is.
Hence, by the "least upper bound principle", it admits a supremum $M = \sup(S)$, i.e. there exists $M \in \mathbb{R}$ satisfying

$$\begin{cases} \forall x \in [a,b),\ F(x) \leq M \\ \forall \varepsilon > 0,\ \exists x_0 \in [a,b),\ M - \varepsilon < F(x_0) \end{cases}$$

We want to show that $\lim\limits_{x\to+\infty} F(x) = M$, i.e.

$$\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in [a,+\infty),\ (x > A \implies |F(x) - M| < \varepsilon)$$

I am just doing the case $b = +\infty$, the other case is quite similar.

Let $\varepsilon > 0$.
We know there exists $x_0 \in [a,+\infty)$ such that $M - \varepsilon < F(x_0)$.
Set $A = x_0$ and let $x \in [a,+\infty)$ satisfying $x > A$.
Since F is non-decreasing, we know that $M - \varepsilon < F(x_0) = F(A) \leq F(x)$. Hence $M - F(x) < \varepsilon$.
But since M is an upper bound of F, we also have that $F(x) \leq M$. Therefore $0 \leq M - F(x) < \varepsilon$ which implies $|F(x) - M| < \varepsilon$.
We proved that $x > A \implies |F(x) - M| < \varepsilon$ as wanted. ■

Theorem 2.2 (The MCT – Part 2: the non bounded case)**.**
Let $F : [a,b) \to \mathbb{R}$ be a function where either $b \in \mathbb{R}_{>a}$ or $b = +\infty$.
If F is non-decreasing and not bounded from above then $\lim\limits_{x\to b^-} F(x) = +\infty$.

Proof. Again, I am just doing the case $b = +\infty$, the other case being quite similar.
We want to prove that $\lim\limits_{x\to b^-} F(x) = +\infty$, i.e.

$$\forall M \in \mathbb{R},\ \exists x_0 \in \mathbb{R},\ \forall x \in [a,b),\ x \geq x_0 \implies F(x) > M$$

Let $M \in \mathbb{R}$. Since f is not bounded from above, there exists $x_0 \in [a,b)$ such that $F(x_0) > M$.
Let $x \in [a,b)$. Assume that $x \geq x_0$, then $M < F(x_0) \leq F(x)$ since F is non-decreasing.
We prove that $x \geq x_0 \implies F(x) > M$ as wanted. ■

Remark 2.3. In one statement, we proved that if $F : [a,b) \to \mathbb{R}$ is a non-decreasing function where either $b \in \mathbb{R}_{>a}$ or $b = +\infty$, then either

- F is bounded from above and then $\lim\limits_{x\to b^-} F(x) = \sup\limits_{[a,b)} F$ (particularly this limit exists), or
- F is not bounded from above and then $\lim\limits_{x\to b^-} F(x) = +\infty$.

3 The BCT and the LCT

Theorem 3.1 (The BCT). *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) satisfying*

(i) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$, and,*

(ii) $\forall x \in [a, b)$, $0 \le f(x) \le g(x)$.

The following statements hold under the above assumptions:

1. *If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)dx$ is divergent.*
2. *If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent.*

Proof.
For $x \in [a, b)$, set $F(x) = \int_a^x f(t)dt$ (which is well-defined since f is integrable on $[a, x]$), then F is non-decreasing: if $x_1 < x_2$ then $F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(t)dt \ge 0$.

Hence, according to the MCT, either F is bounded from above and then $\displaystyle\int_a^b f(t)dt = \lim_{x\to b^-} F(x)$ exists, or it is not bounded from above and then $\displaystyle\lim_{x\to b^-} F(x) = +\infty$ and $\displaystyle\int_a^b f(t)dt$ is divergent.
The same result holds for $G = \int_a^x g(t)dt$.

First case: assume that $\int_a^b f(x)dx$ is divergent.
Since $f(x) \le g(x)$, we have $\int_a^x f(t)dt \le \int_a^x g(t)dt$.
Since the limit of the LHS of the inequality is $+\infty$ (by the above remark), then the limit of the RHS is also $+\infty$.

Second case: assume that $\int_a^b g(x)dx$ is convergent.
Therefore G is bounded from above by some $M \in \mathbb{R}$.
Hence, for any $x \in [a, b)$, $F(x) = \int_a^x f(t)dt \le \int_a^x g(t)dt = G(x) \le M$.
Therefore $F(x)$ is non-decreasing and admits an upper bound.
We deduce from the MCT that $\displaystyle\int_a^b f(x)dx = \lim_{x\to b^-} F(x)$ is convergent. ■

Exercise 3.2. *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) satisfying*

(i) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$, and,*

(ii) $\exists \alpha, \beta > 0$, $\forall x \in [a, b)$, $0 \le \alpha f(x) \le g(x) \le \beta f(x)$.

Prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

Theorem 3.3 (The LCT). *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) satisfying*

(i) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$ (for instance they are continuous),*

(ii) $\forall x \in [a, b)$, $f(x) \ge 0$,

(iii) $\forall x \in [a, b)$, $g(x) > 0$, *and,*

(iv) $\displaystyle\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lambda > 0$ *exists and is positive.*

Then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

Proof. I'm explaining the case $b = +\infty$, the other case is exactly the same.
By definition of the limit (applied to $\varepsilon = \frac{\lambda}{2} > 0$), there exists a $M \in \mathbb{R}$ such that $\forall x \in [a, b)$,

$$x > M \implies \left|\frac{f(x)}{g(x)} - \lambda\right| < \frac{\lambda}{2}$$

We may rewrite the conclusion as

$$\lambda - \frac{\lambda}{2} < \frac{f(x)}{g(x)} < \lambda + \frac{\lambda}{2}$$

which implies that

$$0 < \frac{\lambda}{2} g(x) < f(x) < \frac{3\lambda}{2} g(x)$$

The end of the proof now derives from the above exercise. ■

Remark 3.4. The above results hold for improper integrals of functions of the form $f : (a, b] \to \mathbb{R}$ (i.e. when the integral is improper at the lower bound).
Indeed, if $0 \leq f(x) \leq g(x)$ on $(a, b]$ then $F(x) = \int_x^b f(t)dt$ and $G(x) = \int_x^b g(t)dt$ are non-increasing.
Therefore either F is bounded from above and $\lim_{x \to a^+} F(x)$ exists or $\lim_{x \to a^+} F(x) = +\infty$ (and the same result holds for G).
Hence the above proofs work with slight changes.

Remark 3.5. Notice that $\int_a^b f(x)dx$ is convergent if and only if $\int_a^b -f(x)dx$ is.
Hence, when you want to compare two functions that are both negative, you can multiply them by -1 and then apply the above results.

4 Absolute convergence

Definition 4.1. Let $f : I \to \mathbb{R}$ be a function defined on $I = (a, b]$ (a may be $-\infty$) or $[a, b)$ (b may be $+\infty$).
We say that $\displaystyle\int_a^b f(x)dx$ is *absolutely convergent* if $\displaystyle\int_a^b |f(x)|dx$ is convergent.

The absolute convergence implies the convergence.

Theorem 4.2. *If $\displaystyle\int_a^b f(x)dx$ is absolutely convergent then $\displaystyle\int_a^b f(x)dx$ is convergent.*

Proof. We may assume that $I = [a, b)$, the other cases being similar.
Notice that $\forall x \in I$, $0 \leq f(x) + |f(x)| \leq 2|f(x)|$.
By assumption, $\displaystyle\int_a^b |f(x)|dx$ is convergent, hence by the BCT $\displaystyle\int_a^b (f(x) + |f(x)|)\,dx$ is convergent.
For $c \in [a, b)$, we have $\displaystyle\int_0^c f(x)dx = \int_0^c (f(x) + |f(x)|)dx - \int_a^c |f(x)|dx$.
Then $\displaystyle\lim_{c \to b^-} \int_0^c f(x)dx$ exists since $\displaystyle\lim_{c \to b^-} \int_0^c (f(x) + |f(x)|)dx$ and $\displaystyle\lim_{c \to b^-} \int_0^c |f(x)|dx$ exist. ■

Remark 4.3. The converse is false!!! See the following example!

Example 4.4. Define $f : [1, +\infty) \to \mathbb{R}$ by $f(x) = \frac{\sin(x)}{x}$ then $\displaystyle\int_1^{+\infty} f(x)dx$ is convergent but $\displaystyle\int_1^{+\infty} |f(x)|dx$ is divergent.
Indeed $\displaystyle\int_1^c \frac{\sin(x)}{x}dx = \left[-\frac{\cos(x)}{x}\right]_1^c + \int_1^c \frac{\cos(x)}{x^2}dx = \cos(1) - \frac{\cos(c)}{c} + \int_1^c \frac{\cos(x)}{x^2}dx.$
The last integral is convergent since it is absolutely convergent (use that $0 \leq \left|\frac{\cos(x)}{x^2}\right| \leq \frac{1}{x^2}$) and $\displaystyle\lim_{c \to +\infty} \frac{\cos(c)}{c} = 0$, hence $\displaystyle\int_1^{+\infty} f(x)dx$ is convergent.

Let's prove that f is not absolutely convergent:

$$\begin{aligned}
\int_{\pi}^{(n+1)\pi} \left|\frac{\sin(x)}{x}\right| dx &= \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin(x)}{x}\right| dx \\
&= \sum_{k=1}^{n} \int_{0}^{\pi} \left|\frac{\sin(s+k\pi)}{s+k\pi}\right| ds \\
&\geq \sum_{k=1}^{n} \frac{1}{(k+1)\pi} \int_{0}^{\pi} \sin(s) ds \\
&= \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{(k+1)} \xrightarrow[n\to+\infty]{} +\infty
\end{aligned}$$

5 Some counter-examples

Example 5.1. Let $f : [a, +\infty) \to \mathbb{R}$.
Then $\displaystyle\int_a^{+\infty} f(x)dx$ is convergent does **NOT** imply that $\displaystyle\lim_{x\to+\infty} f(x)$ exists, even if f is non-negative.
Indeed, let f be the function whose graph joins $\left(n - \frac{1}{n^3}, 0\right)$, (n, n) and $\left(n + \frac{1}{n^3}, 0\right)$ by segment lines for $n \in \mathbb{N}_{\geq 2}$ (and 0 otherwise).
Sketch the graph! It helps computing the partial integrals: we sum areas of triangles.
Define $F(x) = \int_0^x f(t)dt$ then F is obviously non-decreasing and

$$F(x) \leq F\left(n + \frac{1}{n^3}\right) = \sum_{k=2}^{n} \frac{1}{2} k \frac{2}{k^3} \leq \sum_{k=1}^{+\infty} \frac{1}{k^2}$$

The last series is convergent. Hence F is non-decreasing and bounded from above. Therefore F admits a limit by the MCT and $\displaystyle\int_a^{+\infty} f(x)dx$ is convergent.
Nevertheless $\displaystyle\lim_{x\to+\infty} f(x)$ doesn't exist.

However, we have the following result:

Exercise 5.2. *If* $\displaystyle\lim_{x\to+\infty} f(x) = \ell$ *exists and* $\displaystyle\int_a^{+\infty} f(x)dx$ *is convergent then* $\ell = 0$.

Example 5.3. Let $f : [a, b) \to \mathbb{R}$.
Then $\displaystyle\int_a^{+\infty} f(x)dx$ is divergent does **NOT** imply that $\displaystyle\lim_{x\to b^-} f(x) = +\infty$, even if f is non-negative.
Indeed, let f the function defined on $(0, 1]$ whose graph joins $\left(\frac{1}{2^p} - \frac{1}{p2^p}, 0\right)$, $\left(\frac{1}{2^p}, 2^p\right)$ and $\left(\frac{1}{2^p} + \frac{1}{p2^p}, 0\right)$ by segment lines for $p \in \mathbb{N}_{\geq 3}$.
Sketch the graph! It helps computing the partial integrals: we sum areas of triangles.
The area of one triangle is $\frac{1}{2} 2 \frac{1}{p2^p} 2^p = \frac{1}{p}$ hence the integral $\int_0^1 f(x)dx$ is divergent since $\displaystyle\sum_{p\geq 1} \frac{1}{p}$ is divergent.
But $\lim_{x\to 0^+} f(x) \neq +\infty$ since $f\left(\frac{1}{2^p} - \frac{1}{p2^p}\right) = 0$.

6 Exercises

Exercise 6.1 (Riemann's integrals)**.**

1. *Prove that* $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha} dx$ *is convergent if and only if* $\alpha > 1$.

2. *Prove that* $\displaystyle\int_{\to 0}^{1} \frac{1}{x^\alpha} dx$ *is convergent if and only if* $\alpha < 1$.

Exercise 6.2. *Let* $f : [a, +\infty) \to \mathbb{R}$ *be a non-negative function which is integrable on any* $[a, c]$ *for* $c > a$.

1. *Prove that if there exists* $\alpha > 1$ *such that* $x^\alpha f(x) \xrightarrow[x\to+\infty]{} 0$ *then* $\displaystyle\int_a^{+\infty} f(x)dx$ *is convergent.*

2. *Prove that if there exists* $\alpha \leq 1$ *such that* $x^\alpha f(x) \xrightarrow[x\to+\infty]{} +\infty$ *then* $\displaystyle\int_a^{+\infty} f(x)dx$ *is divergent.*

Exercise 6.3 (Bertrand's integral)**.**

Prove that $\displaystyle\int_2^{+\infty} \frac{1}{x^\alpha (\ln x)^\beta} dx$ *is convergent if and only if* $\begin{cases} \alpha > 1 \\ \text{or} \\ \alpha = 1 \textit{ and } \beta > 1 \end{cases}$