# Functions of the form $F(x) = \int_S f(x,y)\, dy$

**Example:** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \begin{cases} \dfrac{x^2 y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

Let $x \in \mathbb{R} \setminus \{0\}$ then

$$F(x) := \int_0^1 f(x,y)\, dy = \int_0^1 \frac{x^2 y}{(x^2+y^2)^2}\, dy = \left[ -\frac{x^2}{2(x^2+y^2)} \right]_0^1 = \frac{1}{2(1+x^2)}$$

Let $y \in [0,1]$, then $\lim\limits_{x \to 0} f(x,y) = 0$ (Prove it)

Hence:

$$\underbrace{\lim_{x\to 0} \int_0^1 f(x,y)\, dy}_{= 1/2} \neq \underbrace{\int_0^1 \lim_{x \to 0} f(x,y)\, dy}_{= 0}$$

**Conclusion:** we can NOT simply swap $\int$ and lim

Nevertheless, notice that $f$ is not $C^0$ at $(0,0)$ in the above example since $f(x,x) = \dfrac{1}{4x} \xrightarrow[x \to 0^+]{} +\infty$

If we assume that $f$ is continuous, we have the following result:

**Theorem:** Assume that:

- $S \subset \mathbb{R}^m$ is compact and Jordan measurable (ie $\partial S$ has ZC)
- $T \subset \mathbb{R}^p$ satisfies: $\forall x_0 \in T, \exists r > 0, T \cap \bar{B}(x_0, r)$ is compact
- $f : T \times S \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is continuous

Define $F : T \to \mathbb{R}$ by $F(x) = \int_S f(x,y)\, dy$

Then $F$ is well defined and continuous.

Remark: the condition on $T$ is automatically satisfied if $T$ is

- open: let $x_0 \in T$, since $T$ is open $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subset T$
  set $r = \varepsilon/2$ then $\overline{B}(x_0, r) \cap T = \overline{B}(x_0, r)$ is compact

- closed: $\overline{B}(x_0, r) \cap T$ is closed and bounded hence compact

$\Delta$. Let $x \in T$, then $\begin{array}{l} S \to \mathbb{R} \\ y \mapsto f(x,y) \end{array}$ is $C^0$ on $S$ compact and Jordan measurable so $F(x) = \int_S f(x,y)\, dy$ is well-defined

- If $\int_S 1 = 0$ then $F \equiv 0$ ($F$ is constant equal to $0$), so we may assume that $|S| := \int_S 1 > 0$

- Let's prove that $F$ is $C^0$ at $x_0 \in T$.

Let $\varepsilon > 0$.
By assumption, $\exists r > 0$, $\overline{B}(x_0, r) \cap T$ is compact, hence $f$ is continuous on $\overline{B}(x_0, r) \cap T$ compact and by Heine-Cantor $f : \overline{B}(x_0, r) \cap T \to \mathbb{R}$ is uniformly continuous

so $\exists \delta > 0$, $\forall (x,y), (x',y') \in \overline{B}(x_0, r) \times S$

$$\|(x,y) - (x',y')\| < \delta \Rightarrow |f(x,y) - f(x',y')| < \varepsilon/_{2|S|}$$

Let $x \in T$ satisfying $\|x - x_0\| < \delta' := \min(\delta, r)$

then $x \in \overline{B}(x_0, r)$ and

$\forall y \in S$, $|f(x,y) - f(x_0,y)| < \frac{\varepsilon}{2|S|}$ since $\|(x,y) - (x_0,y)\| = \|x - x_0\| < \delta$

Therefore: $|F(x) - F(x_0)| = \left|\int_S f(x,y) - f(x_0,y)\, dy\right|$
$\leq \int_S |f(x,y) - f(x_0,y)|\, dy \leq \int_S \frac{\varepsilon}{2|S|} = \frac{\varepsilon}{2} < \varepsilon$

We proved: $\forall \varepsilon > 0, \exists \delta' > 0, \forall x \in T, \|x - x_0\| < \delta' \Rightarrow |F(x) - F(x_0)| < \varepsilon$ $\square$

Theorem: Let $S \subset \mathbb{R}^m$ compact and Jordan measurable (ie $\partial S$ has ZC)
$T \subset \mathbb{R}^p$ open
$f: \begin{array}{l} T \times S \to \mathbb{R} \\ (x,y) \mapsto f(x,y) \end{array}$ such that

① $f$ is continuous on $T \times S$

② $\forall i = 1, \dots, p$, $\frac{\partial f}{\partial x_i}$ exists and is continuous on $T \times S$

Then $F: T \to \mathbb{R}$ defined by $F(x) := \int_S f(x,y)\,dy$ is $C^1$

and moreover $\forall i = 1, \dots, p$,

$$\frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\,dy$$

Δ Notice that $F$ is well-defined and continuous by the above theorem

Since $\frac{\partial f}{\partial x_i}$ is $C^0$ on $S$ compact and Jordan measurable,
$\int_S \frac{\partial f}{\partial x_i}$ is well defined

Let $x_0 \in T$, since $T$ is open, for $t$ small enough, $x_0 + t e_i \in T$

We define

$$A(t) := \frac{F(x_0 + te_i) - F(x_0)}{t} - \int_S \frac{\partial f}{\partial x_i}(x_0, y)\,dy \quad \text{for } t \neq 0 \text{ small enough}$$

$$= \int_S \frac{f(x_0 + te_i, y) - f(x_0, y)}{t} - \frac{\partial f}{\partial x_i}(x_0, y)\,dy$$

$$= \int_S \frac{\partial f}{\partial x_i}(x_0 + \theta t, y) - \frac{\partial f}{\partial x_i}(x_0, y)\,dy$$

for some $\theta \in [0,1]$ $\theta(t,y)$ depending on $t$ and $y$, by the MVT (one-variable case)

Let $\varepsilon > 0$.

Since $T$ is open, $\exists r > 0$, $\bar{B}(x_0, r) \subset T$

hence $\frac{\partial f}{\partial x_i} : \bar{B}(x_0, r) \times S \to \mathbb{R}$ is uniformly continuous

as a continuous function on a compact set (Heine-Cantor)

ie $\exists \delta > 0$, $\forall (x,y), (x',y') \in \bar{B}(x_0, r) \times S$

$$\|(x,y) - (x',y')\| < \delta \Rightarrow \left| \frac{\partial f}{\partial x_i}(x,y) - \frac{\partial f}{\partial x_i}(x',y') \right| < \frac{\varepsilon}{2|S|}$$

(I assume $|S| > 0$ otherwise it's easy since $F \equiv 0$

Let $t \in \mathbb{R}$ satisfying $|t| < \min(\delta, r) =: \delta'$

then $\|(x_0, y) - (x_0 + t e_i, y)\| = |t| < \delta$

and $\|x_0 - (x_0 + t e_i)\| = |t| < r$ (ie $x_0 + t e_i \in \bar{B}(x_0, r)$)

so

$$|A(t)| \leq \int_S \frac{\varepsilon}{2|S|} = \frac{\varepsilon}{2} < \varepsilon$$

ie $A(t) \xrightarrow[t \to 0]{} 0$

ie $\frac{F(x_0 + t e_i) - F(x_0)}{t} \xrightarrow[t \to 0]{} \int_S \frac{\partial f}{\partial x_i}(x,y)\, dy$

ie $\frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\, dy$ which is $C^0$ by the previous theorem $\square$

## Summary of this lecture

① Generally $\lim_{x \to x_0} \int_S f(x,y)\,dy \neq \int_S \lim_{x \to x_0} f(x,y)\,dy$

ie we can't swap $\int$ and lim!

② If $f: T \times S \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is $C^0$ (+ technical assumptions on S and T)

then $F: T \to \mathbb{R}$ defined by $F(x) = \int_S f(x,y)\,dy$

is $C^0$

ie we can swap $\int$ and lim

indeed:

$$\lim_{x \to x_0} \int_S f(x,y)\,dy = \lim_{x \to x_0} F(x) \overset{F\ C^0}{=} F(x_0) = \int_S f(x_0,y)\,dy \overset{f\ C^0}{=} \int_S \lim_{x \to x_0} f(x,y)\,dy$$

③ If $f: T \times S \to \mathbb{R}$ is $C^0$, T open, S compact and Jordan measurable

and $\frac{\partial f}{\partial x_i}(x,y)$ exists and is $C^0$ on $\underline{\underline{T \times S}}$

(that's stronger than asking $f_y: x \mapsto f(x,y)$ to be $C^1$)

then F is $C^1$ and we can swap $\int$ and $\frac{\partial}{\partial x_i}$:

$$\frac{\partial \int_S f(x,y)\,dy}{\partial x_i} = \frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\,dy$$

The above theorem can be very useful to compute some integrals that are difficult to compute directly.

Ex: $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$

We introduce $f(t) = \int_0^{2\pi} e^{t\cos\theta} \cos(t\sin\theta)\, d\theta$

then $f'(t) = \int_0^{2\pi} e^{t\cos\theta} \left(\cos\theta \cos(t\sin\theta) - \sin\theta \sin(t\sin\theta)\right) d\theta$

$= 0$ ← If you are in physics, you recognized a line integral, otherwise wait for next week

so $f$ is constant and

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = f(1) = f(0) = \int_0^{2\pi} d\theta = 2\pi$$

Ex: $F(x) = \int_0^x e^{-t^2} dt$, $G(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$

- $F$ is $C^1$ by FTC, $G$ is $C^1$ by the above thm
- $G'(x) = \int_0^1 -2x\, e^{-x^2(1+t^2)} dt \underset{s=xt}{=} -2\int_0^x e^{-x^2-s^2} ds = -2F'(x)F(x) = -(F^2)'$

so $(G+F^2)' = 0$

and $(G+F^2)(x) = (G+F^2)(0) = 0 + \int_0^1 \frac{1}{1+t^2} dt = \frac{\pi}{4}$

Since $\lim_{x\to+\infty} G(x) = 0$

$\int_0^{+\infty} e^{-t^2} dt = \lim_{x\to+\infty} F(x) = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$, ie $\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

(We'll see another proof using polar coordinates on Thursday)