Change of variables formulae

Heuristic

Let $\varphi : J \to I$ be a C^1-diffeomorphism between 2 bounded intervals.

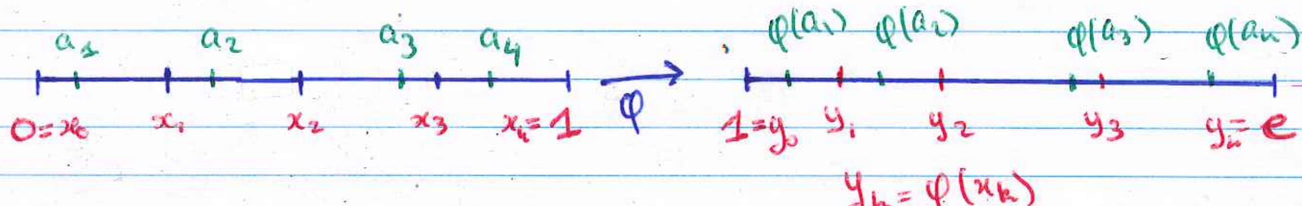
We want to compute $\int_I f$ for $f : I \to \mathbb{R}$ continuous

Since $\varphi : J \to I$ is a bijection between J and I preserving regularity it looks like that we can write $\int_I f = \int_J f \circ \varphi$.

Let's try on an example: $f(x) = x$, $\varphi : [0,1] \to [1,e]$, $\varphi(x) = e^x$

$$\int_{[1,e]} f = \frac{e^2}{2} - \frac{1}{2} \neq \int_{[0,1]} f \circ \varphi = e - 1$$

What went wrong?



A Riemann sum for $f \circ \varphi$ is $f(\varphi(a_1))(x_1 - x_0) + f(\varphi(a_2))(x_2 - x_1) + \cdots$

Notice that it involves $(x_{k+1} - x_k)$ and not $(y_{k+1} - y_k)$ the partition induces by φ on $[1,e]$ from the partition on $[0,1]$.

It doesn't take into account the speed of φ...

We would like to find F on $[0,1]$ s.t.

$$\int_{[0,1]} F = \int_{[1,e]} f$$

At the level of Riemann sums we would like

$$F(a_1)(x_1-x_0)+F(a_2)(x_2-x_1)+F(a_3)(x_3-x_2)+\cdots$$

to be equal to (at least when the step $x_{k+1}-x_k$ goes to 0)

$$f(\varphi(a_1))\,|y_1-y_0|+f(\varphi(a_2))\,|y_2-y_1|+f(\varphi(a_3))\,|y_3-y_2|+\cdots$$

Comment: absolute values because φ could be decreasing

We can set for example

$$F(a_k)(x_{k+1}-x_k)=f(\varphi(a_k))\,|\underset{\varphi(x_{k+1})}{\overset{\shortparallel}{y_{k+1}}}-\underset{\varphi(x_k)}{\overset{\shortparallel}{y_k}}|$$

ie $$F(a_k)=f(\varphi(a_k))\left|\frac{\varphi(x_{k+1})-\varphi(x_k)}{x_{k+1}-x_k}\right|\xrightarrow[x_{k+1}-x_k\to 0]{}f(\varphi(x))\cdot|\varphi'(x)|$$

ie $$\int_I f=\int_J f\circ\varphi\cdot|\varphi'|$$

Remark: in the MAT137 the absolute value was hidden in the fact that $\int_\alpha^\beta=-\int_\beta^\alpha$

So if φ is decreasing ie $\varphi'<0$
$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f\circ\varphi\cdot\varphi'=\int_{\varphi(\beta)}^{\varphi(\alpha)} f\circ\varphi\cdot|\varphi'|$$
with $\varphi(\beta)<\varphi(\alpha)$

We won't be able to use this trick in the multivariable case ie the absolute values are going to be important

Conclusion: $\int_I f\neq\int_J f\circ\varphi$ if $\varphi:J\to I$ is a C^1-diffeomorphism

but $\int_I f=\int_J f\circ\varphi\cdot|\varphi'|$

The above discussion is informal, that's not a proof but it explains well the situation that we are going to clarify now

The one variable case (Recollection from MAT137/MAT157)

Theorem: $I \subset \mathbb{R}$ interval, $\varphi : [a,b] \longrightarrow I$ C^1, $f : I \longrightarrow \mathbb{R}$ C^0

then $$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

Remarks: ① the functions involved are integrable since C^0 on segment lines

that will be a necessary assumption in the multivariable case

② we don't assume φ to be injective

eg: we can compute $\int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} 2t\cos(t^2)\,dt$ using $\varphi(t) = t^2$ even if φ is not injective

③ We may rewrite: $$\int_{[a,b]} f(\varphi(t))\,|\varphi'(t)|\,dt = \int_{\varphi([a,b])} f(x)\,dx$$

when φ is monotonic (ie nondecreasing or nonincreasing)

Proof: Since f is continuous, it admits an antiderivative $F : I \longrightarrow \mathbb{R}$

Notice that $(F\circ\varphi)' = F'\circ\varphi \cdot \varphi' = f\circ\varphi \cdot \varphi'$

Hence
$$\begin{aligned}\int_a^b f\circ\varphi(t)\,\varphi'(t)\,dt &= \int_a^b (F\circ\varphi)'(t)\,dt \\ &= \left[F\circ\varphi\right]_a^b \\ &= \left[F\right]_{\varphi(a)}^{\varphi(b)} \\ &= \int_{\varphi(a)}^{\varphi(b)} F'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx\end{aligned}$$

□

Remarks: It is possible to weaken the assumptions, but the proof becomes more complicated (cf Thm 39 in darboux.pdf)

- Ex (among others): If φ is monotonic, we may simply assume that f is integrable:

ie: If $f : [a,b] \to \mathbb{R}$ is integrable, $\varphi : [c,d] \to \mathbb{R}$ monotonic, $\varphi([c,d]) \subset [a,b]$ and φ' integrable then $\int_{\varphi(c)}^{\varphi(d)} f(x)\,dx = \int_c^d f(\varphi(t))\,\varphi'(t)\,dt$

Mnemonic device

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du \qquad (*)$$

So if you write $u = \varphi(t)$

and then $\dfrac{du}{dt} = \varphi'(t)$

you recover $du = \varphi'(t)\,dt$

⚠ That's just a mnemonic device to remember the proved formula $(*)$

That's not a correct mathematical proof or reasoning

⚠ Do NOT forget to change the bounds:

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du$$

The multivariable case

<u>Theorem</u>: • $U \subset \mathbb{R}^n$ open, $\Phi : U \longrightarrow \mathbb{R}^n$ C^1 injective and $\forall x \in U$, $\det D\Phi(x) \neq 0$

• Let $T \subset U$ be a compact Jordan measurable set (ie ∂T has ZC)

• If f is integrable on $\Phi(T)$ then $f \circ \Phi \cdot |\det D\Phi|$ is integrable on T and

$$\int_{\Phi(T)} f(x)\,dx = \int_T f(\Phi(u))\,|\det D\Phi(u)|\,du$$

<u>Remark</u>: ① We have already seen that for $\Phi : U \to \mathbb{R}^n$ injective (January 16, Transformations)

$$\forall x \in U,\ D\Phi(x) \text{ invertible} \Leftrightarrow \begin{cases} \Phi(U) \text{ is open} \\ \Phi : U \longrightarrow \Phi(U) \text{ is a } C^1 \text{ diffeo} \end{cases}$$

So the condition on Φ is simply that $\Phi : U \longrightarrow V$ is a C^1-diffeomorphism where $V = \Phi(U)$

② $\Phi(T)$ is compact as the continuous image of a compact and $\Phi(T)$ is Jordan measurable (ie $\partial(\Phi(T))$ has ZC)

Δ Idea of proof for ②
- ∂T is closed and bounded, hence compact
- Since Φ is C^1 on ∂T compact, $\exists C > 0$, $\forall x, y \in \partial T$, $\|\Phi(x) - \Phi(y)\| \leq C\|x - y\|$ [See the file "A MVT like inequality" for details]
- Hence $\Phi(\partial T)$ has ZC
- Since Φ is a homeomorphism $\partial(\Phi(T)) = \Phi(\partial T)$
- <u>Ccl</u>: $\partial(\Phi(T))$ has ZC, ie $\Phi(T)$ is Jordan measurable

$\square$

ie we present an idea describing the geometric intuition, it's not a proof!!! ⚠

It is possible to generalize the heuristic idea from before:

$$\int_{\Phi(T)} f(x)\,dx \simeq \sum_S{}' f(\Phi(a_S)) \cdot \mathrm{Vol}(\Phi(S))$$

← (where S is a subrectangle of a partition of a rectangle containing T)

$$\simeq \sum_S{}' f(\Phi(a_S)) \cdot \frac{\mathrm{Vol}(\Phi(S))}{\mathrm{Vol}(S)} \cdot \mathrm{Vol}(S)$$

when $\mathrm{Vol}(S) \to 0$ ⟶

$$\simeq \sum_S{}' f(\Phi(a_S))\, |\det D\Phi(a_S)|\, \mathrm{Vol}(S)$$

$$\simeq \int_T f\circ\Phi \cdot |\det D\Phi|$$

DIFFICULT, NOT MANDATORY:

↳ that's actually the geometric interpretation of the Jacobian determinant:

NOT PART OF MAT157 BUT USEFUL

After a translation, we may assume that $0 \in \mathcal{U}$ and that $\Phi(0) = 0$, and then use the following lemma:

<u>Lemma</u>: $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^n$ two open sets containing 0, $f : \mathcal{U} \to \mathcal{V}$ C^1-diffeo ($0 \mapsto 0$)

$\forall \varepsilon > 0,\ \exists R > 0,\ \forall r \in [0,R],\ (1-\varepsilon)\,Df(0)(B_r) \subset f(B_r) \subset (1+\varepsilon)\,Df(0)(B_r)$

DIFFICULT, DON'T READ IT ⚠

Δ $f(x) - Df(0)x = \eta(x)$ where $\frac{\eta(x)}{\|x\|} \xrightarrow[x\to 0]{} 0$ by differentiability

so $\|f(x) - Df(0)x\| \leqslant C\,\varepsilon'\|x\|$ for x close to 0

$\Rightarrow \|Df(0)^{-1}f(x) - x\| = \|Df(0)^{-1}(f(x) - Df(0)x)\| \leqslant \underbrace{\|Df(0)^{-1}\| \cdot C \cdot \varepsilon'\|x\|}_{\text{take } \varepsilon' \text{ s.t. it is } = \varepsilon}$ for x close to 0 ie in a small ball B_R

$\Rightarrow \|Df(0)^{-1}f(x)\| \leqslant (1+\varepsilon)\|x\|$

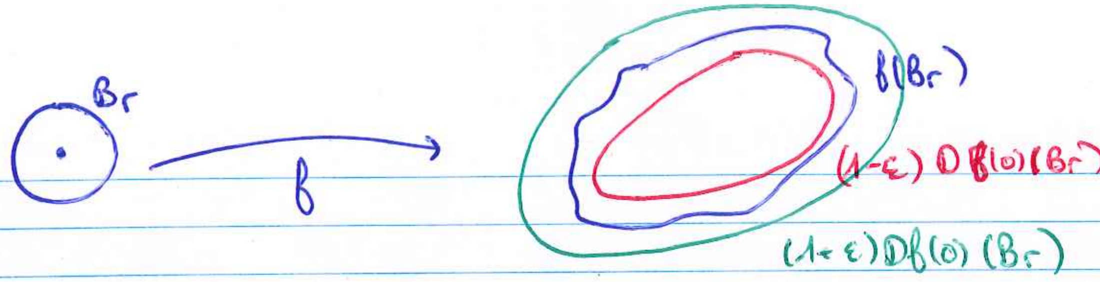
hence the second inclusion

For the first one: $y \overset{\varphi}{\longmapsto} (1-\varepsilon)\,Df(0)y$ is C^0 so $\exists K$ s.t. $\varphi(B(0,K)) \subset \varphi(B_R)$

then if $(1-\varepsilon)\,Df(0)y = f(x)$, $\|x\| - (1-\varepsilon)\|y\| \leqslant \|(1-\varepsilon)y - x\|$
$= \|Df(0)^{-1}(f(x)) - x\| \leqslant \varepsilon\|x\|$

from which we deduce the first inclusion

$\square$



and $\quad \mathrm{Vol}\big((1+\varepsilon)\,Df(0)(B_r)\big) = |\det((1+\varepsilon)\,Df(0))|\ \mathrm{Vol}(B_r)$

$\qquad\qquad\qquad\qquad\qquad\qquad = (1+\varepsilon)^m\,|\det Df(0)|\ \mathrm{Vol}(B_r)$

So $\quad (1-\varepsilon)^m\,|\det Df(0)| \leq \dfrac{\mathrm{Vol}(f(B_r))}{\mathrm{Vol}(B_r)} \leq (1+\varepsilon)^m\,|\det Df(0)|$

END OF THE HEURISTIC IDEA
(which can be fixed to become a formal difficult proof)

Historical comment: the above idea may be fixed and made correct, see for instance Folland

There exists another proof, by induction on m, which is more computational, easier, but hides a little bit the geometric idea.

See for instance Spivak or Munkres

University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Change of variables: usual coordinate systems

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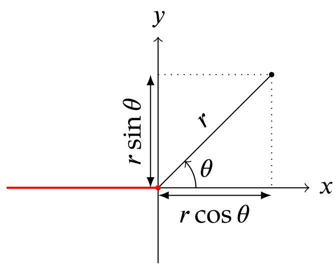
February 13th, 2020

1 Polar coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi) & \to & \mathbb{R}^2 \setminus \{(x,0)\in\mathbb{R}^2 \ :\ x\le 0\} \\ (r,\theta) & \mapsto & (r\cos\theta, r\sin\theta) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta) = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta)| = r$.

Example 1. Let $\Delta = \left\{(x,y)\in\mathbb{R}^2 \ :\ 1\le x^2+y^2\le 9,\ x\ge 0\right\}$.
We want to compute $\displaystyle\int_\Delta e^{x^2+y^2}$.

First, notice that $\Delta = \Phi\left([1,3]\times[-\pi/2,\pi/2]\right)$.
Hence,

$$\begin{aligned}
\int_\Delta e^{x^2+y^2}\,dx\,dy &= \int_{[1,3]\times[-\pi/2,\pi/2]} e^{r^2} r\,dr\,d\theta && \text{by the CoV formula} \\
&= \int_{-\pi/2}^{\pi/2}\int_1^3 re^{r^2}\,dr\,d\theta && \text{by the iterated integrals theorem} \\
&= \int_{-\pi/2}^{\pi/2} \frac{e^9-e}{2}\,d\theta \\
&= \frac{\pi}{2}\left(e^9-e\right)
\end{aligned}$$

Example 2. We want to compute $\displaystyle\int_{\overline{B}((1,1),1)} x^2+y^2-2y\,dx\,dy.$

First notice that $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Psi(x,y) = (x+1, y+1)$ is a C^1-diffeomorphism and that $|\det D\Psi(x,y)| = 1$.
Moreover $\overline{B}((1,1),1) = \Psi\left(\overline{B}(\mathbf{0},1)\right)$. Hence

$$\begin{aligned}\int_{\overline{B}((1,1),1)} x^2+y^2-2y\,dx\,dy &= \int_{\overline{B}(\mathbf{0},1)} (x+1)^2+(y+1)^2-2(y+1)\,dx\,dy && \text{by the CoV formula}\\ &= \int_{\overline{B}(\mathbf{0},1)} x^2+y^2-2x\,dx\,dy\end{aligned}$$

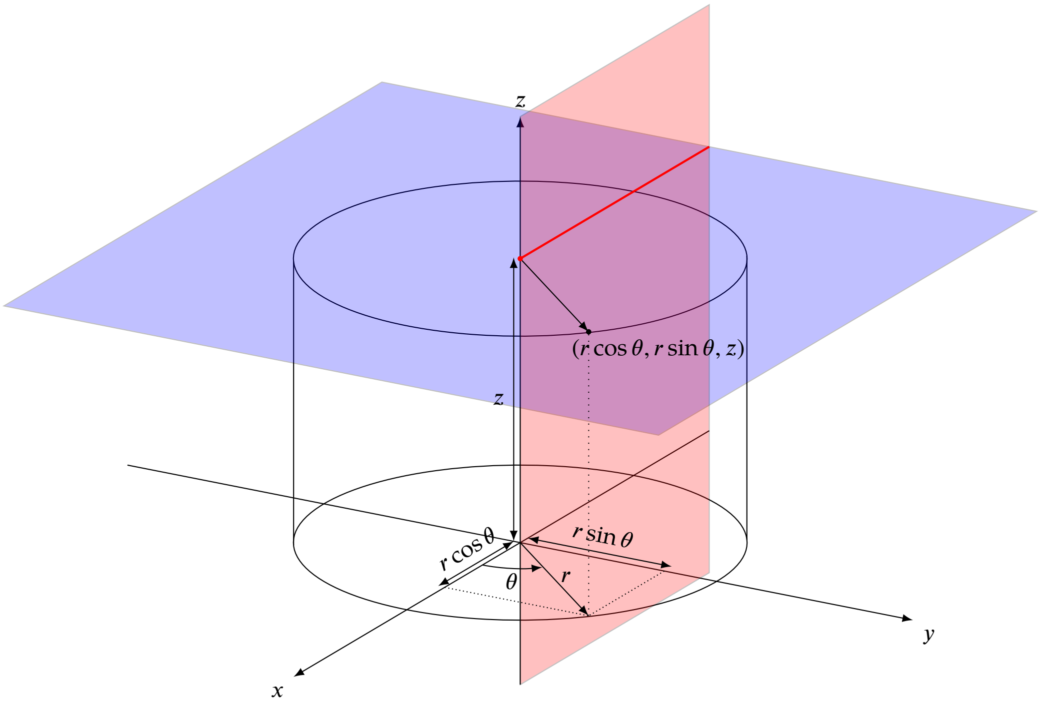
Next, we have $\overline{B}(\mathbf{0},1) = \Phi([0,1]\times[-\pi,\pi])$.
Notice that there is an issue for $r=0$ or $\theta=\pm\pi$ (i.e. $\{(x,0) : x\in[-1,0]\}$) but these sets have zero content.
Hence

$$\int_{\overline{B}(\mathbf{0},1)} x^2+y^2-2x\,dx\,dy = \int_{[0,1]\times[-\pi,\pi]} (r^2-2r\cos\theta)r\,dr\,d\theta \qquad \text{by the CoV formula}$$

$$\begin{aligned}&= \int_{-\pi}^{\pi}\int_0^1 r^3-2r^2\cos\theta\,dr\,d\theta && \text{by the iterated integrals theorem}\\ &= \int_{-\pi}^{\pi} \frac{1}{4}-\frac{2}{3}\cos\theta\,d\theta\\ &= \frac{\pi}{2}\end{aligned}$$

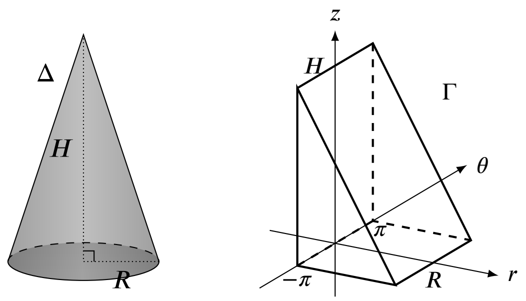
2 Cylindrical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi)\times\mathbb{R} & \to & \mathbb{R}^3 \setminus ((-\infty,0]\times\{0\}\times\mathbb{R}) \\ (r,\theta,z) & \mapsto & (r\cos\theta, r\sin\theta, z) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta,z) = \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,z)| = r$.

Example 3. We want to compute $\displaystyle\int_\Delta z dx dy dx$
where $\Delta = \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 \le \frac{R^2}{H^2}(H - z)^2,\, 0 \le z \le H\}$.



Notice that $\Delta = \Phi(\Gamma)$ where $\Gamma = \left\{(r, \theta, z) \;:\; 0 \le r \le \frac{R}{H}(H - z),\, \theta \in [-\pi, \pi],\, z \in [0, H]\right\}$.
Again, Γ goes outside the domain of Φ but the involved sets have zero content.
Hence

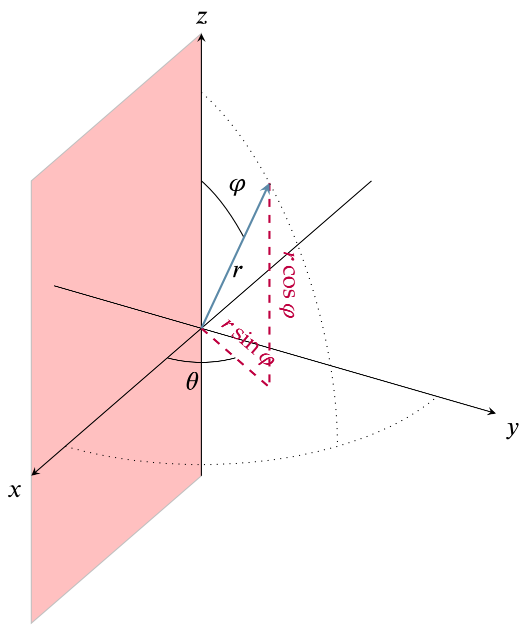
$$\begin{aligned}
\int_\Delta z dx dy dx &= \int_\Gamma z r dr d\theta dz \qquad \text{by the CoV formula}\\
&= \int_0^H \int_{-\pi}^{\pi} \int_0^{\frac{R}{H}(H-z)} r z dr d\theta dz\\
&= \int_0^H \int_{-\pi}^{\pi} \frac{R^2}{2H^2}(H - z)^2 z d\theta dz\\
&= \frac{\pi R^2}{H^2} \int_0^H (H - z)^2 z dz\\
&= \frac{\pi R^2 H^2}{12}
\end{aligned}$$

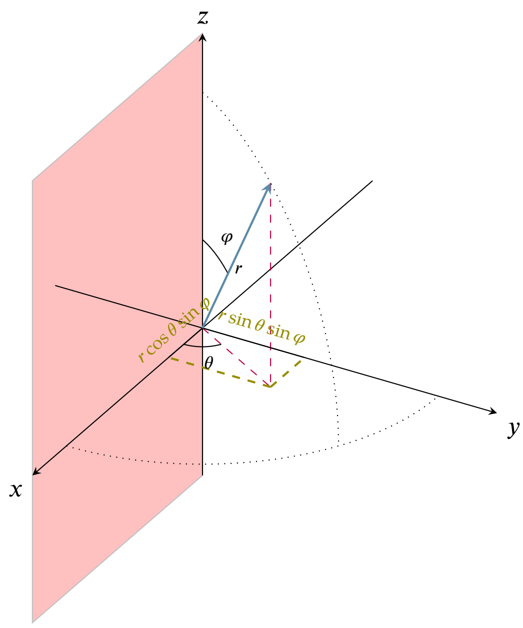
3 Spherical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(0,2\pi)\times(0,\pi) & \to & \mathbb{R}^3 \setminus ([0,+\infty)\times\{0\}\times\mathbb{R}) \\ (r,\theta,\varphi) & \mapsto & (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \end{array}$$

In this course, we use the following convention [⋆]:

$$(r,\theta,\varphi) = (\text{radius/distance to the origin}, \text{longitude}, \text{colatitude})$$





- Φ is C^1.
- Φ is bijective.
- The Jacobian determinant is

$$\begin{aligned}
\det D\Phi(r,\theta,\varphi) &= \det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix} \\
&= \cos\varphi\det\begin{pmatrix} -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} - r\sin\varphi\det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi \end{pmatrix} \\
&= r^2\cos^2\varphi\sin\varphi\det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} - r^2\sin^3\varphi\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= -r^2\cos^2\varphi\sin\varphi - r^2\sin^3\varphi \\
&= -r^2\sin\varphi < 0 \text{ since } \varphi\in(0,\pi)
\end{aligned}$$

- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,\varphi)| = r^2\sin\varphi$.

[⋆] This convention may differ from the one used in other courses in math or in physics (the meaning of θ and φ may be swapped, some people use the latitude and not the colatitude...).
I believe that the usual convention in physics is (r,θ,φ) = (radius, colatitude, longitude) as in ISO 80000-2, i.e. the meaning of θ and φ are swapped from our convention in MAT237.

Example 4. We want to compute $\int_\Delta z dx dy dz$ where $\Delta = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 \leq 1, \ z \geq 0\right\}$.
Notice that $\Delta = \Phi\left([0,1]\times[0,2\pi]\times[0,\pi/2]\right)$.
Again, there is an issue with the domain of Φ but the involved sets have zero content.
Hence

$$\begin{aligned}
\int_\Delta z dx dy dz &= \int_{[0,1]\times[0,2\pi]\times[0,\pi/2]} r^3 \cos\varphi \sin\varphi dr d\theta d\varphi && \text{by the CoV formula} \\
&= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^3 \frac{\sin(2\varphi)}{2} dr d\theta d\varphi && \text{by the iterated integrals theorem} \\
&= 2\pi \frac{1}{4}\left(\frac{\cos 0}{4} - \frac{\cos\pi}{4}\right) \\
&= \frac{\pi}{4}
\end{aligned}$$