Iterated integrals

Theorem (Fubini's theorem)

Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ be two rectangles

Let $f : A \times B \to \mathbb{R}$ be integrable ⚠

Define $l, u : A \longrightarrow \mathbb{R}$ by

$$l(x) := \underline{\int_B} f(x,y)\,dy$$

$$u(x) := \overline{\int_B} f(x,y)\,dy$$

For $x \in A$ fixed, $f_x : \begin{array}{l} B \longrightarrow \mathbb{R} \\ y \longmapsto f(x,y) \end{array}$ is bounded.
Hence $\underline{\int_B} f_x$ and $\overline{\int_B} f$ are well defined.
However there is no reason for f_x to be integrable
ie: it is possible that $l(x) \neq u(x)$

Then l and u are integrable and

$$\int_{A \times B} f = \int_A l(x)\,dx = \int_A \left(\underline{\int_B} f(x,y)\,dy \right) dx$$

$$\int_{A \times B} f = \int_A u(x)\,dx = \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx$$

⚠ **Remark**: it is possible for $f_x : \begin{array}{l} B \longrightarrow \mathbb{R} \\ y \longmapsto f(x,y) \end{array}$ to not be integrable

ie the lower/upper integral is important, we can't omit it!

Ex: $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$, $f(x,y) = \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

then $\underline{\int_B} f(1/2, y)\,dy = 0 \neq \overline{\int_B} f(1/2, y)\,dy$ so $f_{1/2}$ is not integrable

Nevertheless f is integrable and: $0 = \int_{A \times B} f = \int_A \left(\underline{\int_B} f(x,y)\,dy \right) dx$
$= \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx$

Ex: even worse: $f : [0,1]\times[0,1] \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q},\ y \notin \mathbb{Q} \\ 1 - 1/q & \text{if } x = p/q,\ \gcd(p,q)=1,\ p,q \in \mathbb{N}_{\geq 0}, q \neq 0,\ y \in \mathbb{Q} \end{cases}$$

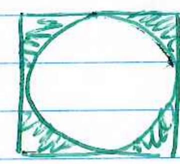
$$\int_{[0,1]\times[0,1]} f = 1$$

$$\int_0^1 f(x,y)\,dy = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ \text{DNE} & \text{if } x \in \mathbb{Q} \end{cases}$$

Ex: how to use Fubini's theorem to compute an integral

$$C = \left\{ (x,y) \in [-1,1]^2 : \|(x,y)\| \geq 1 \right\}$$

C



$$\int_C f = \int_{[-1,1]\times[-1,1]} \chi_C f$$

$$= \int_{-1}^1 \left(\int_{-1}^1 f(x,y)\,\chi_C(x,y)\,dy \right) dx$$

$$= \int_{-1}^1 \left(\int_{-1}^{-\sqrt{1-x^2}} f(x,y)\,dy + \int_{\sqrt{1-x^2}}^1 f(x,y)\,dy \right) dx$$

since $\chi_C(x,y) = \begin{cases} 1 & \text{if } y \geq \sqrt{1-x^2} \text{ or } y \leq -\sqrt{1-x^2} \\ 0 & \text{otherwise} \end{cases}$

I do the proof for ℓ only, the proof for u is similar

Δ Let $\varepsilon > 0$
Then there exists a partition P of $A \times B$ s.t. $U_P(f) - L_P(f) < \varepsilon$

P induces a partition P_A of A and a partition P_B of B
(a subrectangle S of P is of the form $S_A \times S_B$)

$$L_P(f) = \sum_S{}' \partial(S) \inf_S (f)$$
$$= \sum_{S_A, S_B}{}' \partial(S_A \times S_B) \inf_{S_A \times S_B} (f)$$

$$(*) \quad L_P(f) = \sum_{S_A}{}' \left(\sum_{S_B}{}' \inf_{S_A \times S_B} (f)\, \partial(S_B) \right) \partial(S_A)$$

If S_A is fixed and $x \in S_A$, then:

$$\sum_{S_B}{}' \inf_{S_A \times S_B} (f)\, \partial(S_B) \leqslant \sum_{S_B}{}' \inf_{y \in S_B} (f(x,y))\, \partial(S_B)$$
$$\leqslant \underline{\int}_B f(x,y)\, dy = \ell(x)$$

(We shrink the domain:
$\{x\} \times S_B \subset S_A \times S_B$
$\Rightarrow \inf_{S_A \times S_B} f \leqslant \inf_{\{x\} \times S_B} f$)

So $\displaystyle\sum_{S_B}{}' \inf_{S_A \times S_B} (f)\, \partial(S_B) \leqslant \inf_{S_A} \ell(x)$

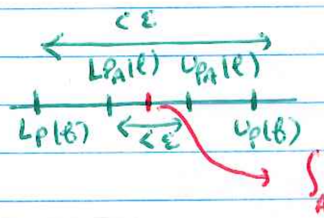
Hence $(*)$ gives:

$$L_P(f) = \sum_{S_A}{}' \left(\sum_{S_B}{}' \inf_{S_A \times S_B} (f)\, \partial(S_B) \right) \partial(S_A) \leqslant \sum_{S_A}{}' \left(\inf_{S_A} \ell(x) \right) \partial(S_A) = L_{P_A}(\ell)$$

ie $L_P(f) \leqslant L_{P_A}(\ell) \quad (**)$

and similarly, we could prove $U_{P_A}(u) \leqslant U_P(f) \quad (***)$

So: $\underset{(**)}{L_P(f) \leqslant L_{P_A}(\ell)} \underset{\text{always true}}{\leqslant} U_{P_A}(\ell) \underset{\text{since } \ell \leq u}{\leqslant} U_{P_A}(u) \underset{(***)}{\leqslant} U_P(f)$



then $U_{P_A}(\ell) - L_{P_A}(\ell) \leqslant U_P(f) - L_P(f) < \varepsilon$

So ℓ is integrable and moreover: $\displaystyle\int_{A \times B} f = \int_A \ell$

$\square$

Corollary: Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ be two rectangles
Let $f : A \times B \to \mathbb{R}$

If (i) $f : A \times B \longrightarrow \mathbb{R}$ is integrable

(ii) $\forall x \in A$, $f_x : B \to \mathbb{R}$ defined by $f_x(y) = f(x,y)$ is integrable

Then

(1) $g : A \longrightarrow \mathbb{R}$ defined by $g(x) = \int_B f_x(y)\,dy = \int_B f(x,y)\,dy$ is integrable.

(2) $\int_{A \times B} f = \int_A g = \int_A \left(\int_B f(x,y)\,dy \right) dx$

Δ (ii) ensures that $g = u = l$ in Fubini's theorem ◻

Corollary. $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ rectangles, $f : A \times B \longrightarrow \mathbb{R}$ continuous

then $\int_{A \times B} f = \int_A \left(\int_B f(x,y)\,dy \right) dx$

Δ f and f_x are continuous and hence integrable and we may apply the above corollary ◻

⚠ We can NOT weaken the assumption to assume that the discontinuity set has ZC

$$f : [0,1] \times [0,1] \longrightarrow \mathbb{R}, \quad f(x,y) = \begin{cases} 1 & \text{if } x = 1/2,\ y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

The discontinuity set of f has ZC but the discontinuity set of $f_{1/2} : y \mapsto f(1/2, y)$ doesn't have ZC and $\int_{[0,1]} f(1/2, y)\,dy$ is not defined.