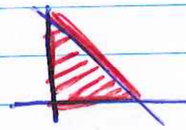
Ex 1

① • $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy(1-x-y)$ is continuous



• $K = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$

→ K is bounded : for $(x,y) \in K$: $0 \leq x \leq x+y \leq 1 \Rightarrow x \in [0,1]$
$0 \leq y \leq x+y \leq 1 \Rightarrow y \in [0,1]$

so $K \subset [0,1] \times [0,1]$ and K is bounded

→ K is closed:
$K = \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \cap \{(x,y) \in \mathbb{R}^2 : y \geq 0\} \cap \{(x,y) \in \mathbb{R}^2 : x+y \leq 1\}$
each of these sets is closed as the inverse image of a closed set by a continuous function
eg: $\{(x,y) \in \mathbb{R}^2 : x+y \leq 1\} = \varphi^{-1}((-\infty, 1])$, $\varphi(x,y) = x+y$

So K is closed as a finite intersection of closed sets

Ccl: K is compact as a bounded + closed set

• $f : K \to \mathbb{R}$ has a max and a min as a continuous function defined on a compact set.

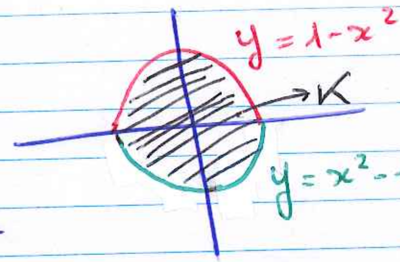
• Let's study f on $\mathring{K} = \{x > 0, y > 0, x+y < 0\}$ open (So we can use the first derivative test

$Df(x,y) = (y(1-2x-y), x(1-x-2y))$
notice that $(x,y) \in \mathring{K} \Rightarrow x \neq 0$ and $y \neq 0$
so a critical point on $\mathring{K}$ satisfies $\begin{cases} 1-2x-y = 0 \\ 1-x-2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1/3 \\ y = 1/3 \end{cases}$

• on ∂K , $f(x,y) = 0$ Δ $\partial K = \{(x,0) : x \in [0,1]\} \cup \{(0,y) , y \in [0,1]\}$
$\cup \{(x, 1-x) : x \in [0,1]\}$

• Since f admits a min and a max,
the max has to be $f(1/3, 1/3) = 1/27$ at $(1/3, 1/3)$
the min has to be 0 on ∂K

② $\cdot$ f is continuous on $\mathbb{R}^2$



- $K = \{(x,y) \in \mathbb{R}^2 : x^2-1 \leq y \leq 1-x^2\}$

<u>K is bounded</u>: $(x,y) \in K \Rightarrow x^2-1 \leq 1-x^2$
$\Rightarrow x^2 \leq 1$
$\Rightarrow x \in [-1,1]$

so $-1 \leq x^2-1 \leq y \leq 1-x^2 \leq 1 \Rightarrow y \in [-1,1]$

hence $K \subset [-1,1] \times [-1,1]$ is bounded.

<u>K is closed</u>: $K = \{(x,y) \in \mathbb{R}^2 : x^2-1-y \leq 0\} \cap \{(x,y) \in \mathbb{R}^2 : 0 \leq 1-x^2-y\}$ is closed as the intersection of two closed sets (each being closed as the C^0- inverse image of a closed subset of $\mathbb{R}$)

<u>Ccl</u>: K is compact

- Hence $f : K \to \mathbb{R}$ admits a min and a max as a C^0 function defined on a compact set

- We study f on $\mathring{K}$ open:

$\nabla f(x,y) = (2x(1-y), 2y-x^2)$

So $\nabla f(x,y) = 0 \Leftrightarrow (x,y) = (0,0)$ or $(-\sqrt{2},1)$ or $(\sqrt{2},1)$

only $(0,0) \in \mathring{K}$ and $f(0,0) = 0$

- We study f on $\partial K = \{(x,1-x^2) : x \in [-1,1]\} \cup \{(x,x^2-1) : x \in [-1,1]\}$

$g(x) = f(x, 1-x^2) = 1-2x^2+2x^4$

x	-1		$-1/\sqrt{2}$		0		$1/\sqrt{2}$		1
g	1	↘	$1/2$	↗	1	↘	$1/2$	↗	1

$h(x) = f(x, x^2-1) = 1$

- <u>Ccl</u>: the min has to be 0

the max has to be 1

(3) $f(x,y,z) = x + 2y + 3z$ is C^0 on $\mathbb{R}^3$

$K = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\ y - z = 2\}$

$\rightarrow (x,y,z) \in K \Rightarrow x^2 \leq x^2+y^2 = 1 \Rightarrow x \in [-1,1]$

and similarly $y \in [-1,1]$

so $z = y - 2 \in [-3,-1]$

Hence $K \subset [-1,1] \times [-1,1] \times [-3,-1]$ is bounded

$\rightarrow K = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\} \cap \{(x,y,z) \in \mathbb{R}^3 : y-z=2\}$
is closed as the intersection of two closed sets
(each one being closed as the inverse image of a closed set of $\mathbb{R}$ by a C^0 function)

Ccl: K is compact as a closed and bounded set
and $f : K \to \mathbb{R}$ has a min and a max as a continuous function on a compact set

By Lagrange multipliers theorem, if $(x,y,z) \in K$ is a local extremum of $f : K \to \mathbb{R}$ then $\exists \lambda, \mu \in \mathbb{R}$ s.t.

$$\nabla f(x,y,z) = \lambda \nabla g_1(x,y,z) + \mu \nabla g_2(x,y,z)$$

where $g_1(x,y,z) = x^2+y^2$
$g_2(x,y,z) = y - z$

ie $$\begin{pmatrix}1\\2\\3\end{pmatrix} = \lambda \begin{pmatrix}2x\\2y\\0\end{pmatrix} + \mu \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

By the last row $\mu = -3$ and then from the first two rows:

$$\begin{cases} 2x\lambda = 1 \\ 2y\lambda = 5 \end{cases} \Rightarrow \begin{cases} x = 1/2\lambda \\ y = 5/2\lambda \end{cases}, \text{ but } x^2+y^2=1 \text{ so } \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 1$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{13}{2}}$$

So $(x,y,z) = (x,y,y-2) = \left(\frac{1}{2\lambda}, \frac{5}{2\lambda}, \frac{5}{2\lambda} - 2\right) = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}, \frac{5}{\sqrt{26}} - 2\right)$ or $\left(-\frac{1}{\sqrt{26}}, \frac{-5}{\sqrt{26}}, \frac{-5}{\sqrt{26}} - 2\right)$

$$\underbrace{f\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}, \frac{5}{\sqrt{26}} - 2\right) = \sqrt{26} - 6}_{\text{max}} > \underbrace{-\sqrt{26} - 6}_{\text{min}} = f\left(\frac{-1}{\sqrt{26}}, \frac{-5}{\sqrt{26}}, \frac{-5}{\sqrt{26}} - 2\right)$$

of f on K

<u>Ex 2</u>: we want to minimize $f(x,y) = x^2 + y^2$ on $L = \{(x,y) \in \mathbb{R}^2 : 2x - y = 16\}$

Notice that $L = \{(x, 2x-16) : x \in \mathbb{R}\}$

so we study $g(x) = f(x, 2x-16) = x^2 + (2x-16)^2$

$$= x^2 + 4x^2 - 64x + 256$$

$$= 5x^2 - 64x + 256$$

$g'(x) = 10x - 64$, $g'(x) = 0 \Leftrightarrow x = \frac{64}{10} = \frac{32}{5}$

	$-\infty$		$\frac{32}{5}$		$+\infty$
g'		$-$	0	$+$	
g	$+\infty$	$\searrow$		$\nearrow$	$+\infty$

$$g(32/5) = 5\left(\frac{32}{5}\right)^2 - \frac{64 \times 32}{5} + 256$$

$$= \frac{1024}{5} - \frac{2048}{5} + 256 = \frac{256}{5}$$

So the min of f on L is $\frac{256}{5}$

at $\left(\frac{32}{5}, 2 \cdot \frac{32}{5} - 16\right) = \left(\frac{32}{5}, -\frac{16}{5}\right)$

<u>CCL</u>: The closest point to the origin on the line L

is $\left(\frac{32}{5}, -\frac{16}{5}\right)$

Ex 3. $F(x,y) = x^4 + x^3y^2 - y + y^2 + y^3$ is C^1 on $\mathbb{R}^2$

(we can't apply the ImFT if F is diff not C^1)

$$\frac{\partial F}{\partial y}(x,y) = 2x^3y - 1 + 2y + 3y^2$$

$$\frac{\partial F}{\partial y}(-1,1) = 2 \neq 0$$

$$F(-1,1) = 1$$

Hence, by the implicit function theorem, in a neighborhood of $(-1,1)$, the level set $\{(x,y) \in \mathbb{R}^2 : F(x,y) = 1\}$ is described by a C^1 graph $y = \varphi(x)$ (and $\varphi(-1)=1$)

Computation of $\frac{\partial \varphi}{\partial x}(-1)$:

Method 1: $F(x, \varphi(x)) = 1$

Chain rule $\Rightarrow \frac{\partial F}{\partial x}(x,\varphi(x)) + \frac{\partial \varphi}{\partial x}(x) \cdot \frac{\partial F}{\partial y}(x,\varphi(x)) = 0$

set $x=-1$
remember $\varphi(-1)=1$ $\Rightarrow \frac{\partial F}{\partial x}(-1,1) + \frac{\partial \varphi}{\partial x}(-1) \cdot \frac{\partial F}{\partial y}(-1,1) = 0$

$$\Rightarrow \frac{\partial \varphi}{\partial x}(-1) = -\frac{\frac{\partial F}{\partial x}(-1,1)}{\frac{\partial F}{\partial y}(-1,1)} = -\frac{-1}{2} = \boxed{\frac{1}{2}}$$

Method 2: $F(x,\varphi(x)) = 1 \iff x^4 + x^3\varphi(x)^2 - \varphi(x) + \varphi(x)^2 + \varphi(x)^3 = 1$

differentiate $\Rightarrow$ $4x^3 + 3x^2\varphi(x)^2 + 2x^3\varphi(x)\varphi'(x) - \varphi'(x) + 2\varphi(x)\varphi'(x) + 3\varphi'(x)\varphi(x)^2 = 0$

not a good idea if F is not simple

set $x=-1$
so $\varphi(x) = 1$ $\Rightarrow -4 + 3 - \cancel{2\varphi'(-1)} - \varphi'(-1) + \cancel{2}\varphi'(-1) + 3\varphi'(-1) = 0$

$$\Rightarrow -1 + 2\varphi'(-1) = 0 \Rightarrow \varphi'(-1) = \boxed{1/2}$$

Ex 4 : $F(x,y,z) = x + y + z + \sin(xyz)$. is C^1 on $\mathbb{R}^3$

Method 1 : it's not useful to compute $DF(0,0,0)$ for the ImFT, but will be useful for $D\varphi(0)$

with the Jacobian

$$DF(x,y,z) = \left(1 + yz\cos(xyz) \quad 1 + xz\cos(xyz) \quad 1 + yz\cos(yz)\right)$$

$$DF(0,0,0) = (1,\ 1,\ 1)$$

$\hookrightarrow D_yF(0,0,0) = \frac{\partial F}{\partial y}(0,0,0)$

$F(0,0,0) = 0$

So $D_yF(0,0,0) = 1 \neq 0$ and $\{F(x,y,z) = 0\}$ is locally around $(0,0,0)$ a C^1 graph $y = \varphi(x,z)$ by the implicit function theorem

$G(x,z) = F(x, \varphi(x,z), z) = 0$

$$\underbrace{\vec{0}}_{\text{constant}} = DG(0,0) \overset{\text{chain rule}}{=} DF(0,\overset{0}{\overbrace{\varphi(0,0)}},0) \cdot \begin{pmatrix} 1 & 0 \\ \frac{\partial \varphi}{\partial x}(0,0) & \frac{\partial \varphi}{\partial z}(0,0) \\ 0 & 1 \end{pmatrix}$$

$$= (1\ 1\ 1) \begin{pmatrix} 1 & 0 \\ \frac{\partial \varphi}{\partial x}(0,0) & \frac{\partial \varphi}{\partial z}(0,0) \\ 0 & 1 \end{pmatrix}$$

$$(0,0) = \left(1 + \frac{\partial \varphi}{\partial x}(0,0) \quad \frac{\partial \varphi}{\partial z}(0,0) + 1\right)$$

So $\frac{\partial \varphi}{\partial x}(0,0) = -1$, $\frac{\partial \varphi}{\partial z}(0,0) = -1$ and $D\varphi(0,0) = (-1\ -1)$

or you use the formula from the lecture:

$$D\varphi(0,0) = -\left[D_yF(0,0,0)\right]^{-1} \cdot D_{xz}F(0,0,0)$$

⚠ don't forget it

$$= -\ [1]^{-1}\ [1\ 1]$$

$$D\varphi(0,0) = -\ [1\ 1] = (-1, -1)$$

$\frac{\partial \varphi}{\partial x}(0,0)$ $\qquad$ $\frac{\partial \varphi}{\partial y}(0,0)$

F is C^1 on $\mathbb{R}^3$ (otherwise you can't apply the Imp FT)

Method 2: partial derivatives no Jacobian

$$\frac{\partial F}{\partial y}(x,y,z) = 1 + xz\cos(xyz)$$

$$\frac{\partial F}{\partial y}(0,0,0) = 1 \neq 0 \quad \text{and} \quad F(0,0,0) = 0$$

Hence the level set $\{F(x,y,z)=0\}$ is locally around $(0,0,0)$ the graph of a C^1 function $y = \varphi(x,z)$

$$F(x,\varphi(x,z),z) = 0$$

Chain rule to compute $\partial/\partial x$

$$\Rightarrow \underbrace{1}_{\frac{\partial x}{\partial x}} \cdot \frac{\partial F}{\partial x}(x,\varphi(x,z),z) + \frac{\partial \varphi}{\partial x}(x,z)\cdot\frac{\partial F}{\partial y}(x,\varphi(x,z),z) + \underbrace{0}_{\frac{\partial z}{\partial x}}\,\frac{\partial F}{\partial z}(x,\varphi(x,z),z) = 0$$

$x=z=0 \Rightarrow \varphi(x,z)=0$

$$\Rightarrow \frac{\partial F}{\partial x}(0,0,0) + \frac{\partial \varphi}{\partial x}(0,0)\cdot\frac{\partial F}{\partial y}(0,0,0) = 0$$

$$\Rightarrow 1 + \frac{\partial \varphi}{\partial x}(0,0) = 0$$

$$\Rightarrow \boxed{\frac{\partial \varphi}{\partial x}(0,0) = -1}$$

Chain rule for $\partial/\partial z$:

$$\underbrace{0}_{\frac{\partial x}{\partial z}} \cdot \frac{\partial F}{\partial x}(x,\varphi(x,z),z) + \frac{\partial \varphi}{\partial z}(x,z)\cdot\frac{\partial F}{\partial y}(x,\varphi(x,z),z) + \underbrace{1}_{\frac{\partial z}{\partial z}}\,\frac{\partial F}{\partial z}(x,\varphi(x,z),z) = 0$$

$$x=z=0 \quad \Rightarrow \frac{\partial \varphi}{\partial z}(0,0)\cdot\frac{\partial F}{\partial y}(0,0,0) + \frac{\partial F}{\partial z}(0,0,0) = 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial z}(0,0) + 1 = 0$$

$$\Rightarrow \boxed{\frac{\partial \varphi}{\partial z}(0,0) = -1}$$

or you use directly the formula from class:

$$\frac{\partial \varphi}{\partial x}(0,0) = -\frac{\frac{\partial F(0,0,0)}{\partial x}}{\frac{\partial F(0,0,0)}{\partial y}} = -\frac{1}{1} = -1$$

Ex 5: $F(x,y,z) = (x^2 - y^2 + 2z^2, xyz)$ is C^1 on $\mathbb{R}^3$
and $F(1,1,1) = (2,1)$

$$DF(x,y,z) = \begin{pmatrix} 2x & -2y & 4z \\ yz & xz & xy \end{pmatrix}$$

$$DF(1,1,1) = \begin{pmatrix} 2 & -2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$
$\partial/\partial x$ $\quad$ $\partial/\partial z$

← It's not useful to compute DF entirely for the ImpFT, it's enough to compute $D_{xz}F$. But DF will be useful later for $D\varphi$

$D_{\underline{xz}}F(1,1,1) = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ is invertible $\left(\det\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} = 2-4 = -2 \neq 0\right)$

Hence, by the implicit function theorem, the level set $\{F(x,y,z) = (2,1)\}$ is locally around $(1,1,1)$ the graph of a C^1 function $\underline{(x,z)} = \varphi(y)$.

$$G(y) = F(\varphi_1(y), y, \varphi_2(y)) = (2,1)$$

$$\Rightarrow \vec{0} = DG(1) = DF(\varphi_1(1), 1, \varphi_2(1)) \cdot \begin{pmatrix} \frac{\partial \varphi_1(1)}{\partial y} \\ 1 \\ \frac{\partial \varphi_2}{\partial y}(1) \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{2} & -2 & \boxed{4} \\ \boxed{1} & 1 & \boxed{1} \end{pmatrix} \begin{pmatrix} \boxed{\partial\varphi_1/\partial y(1)} \\ 1 \\ \boxed{\partial\varphi_2/\partial y(1)} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \partial\varphi_1/\partial y(1) \\ \partial\varphi_2/\partial y(1) \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot (1)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} D\varphi(1) + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow D\varphi(1) = -\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \begin{matrix} \to \frac{\partial \varphi_1}{\partial y}(1) \\ \to \frac{\partial \varphi_2}{\partial y}(1) \end{matrix}$$

Or you directly apply the formula from class:

don't forget it

$$D\varphi(1) = -\left[D_{xz}F(1,1,1)\right]^{-1}\left[D_yF(1,1,1)\right] = -\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

↳ xz in terms of y

Exo 6: 1 • f is differentiable and hence continuous

• assume by contradiction that f is not injective

i.e. $\exists x, y \in \mathbb{R}$, $x \neq y$ and $f(x) = f(y)$

then by Rolle's theorem $\exists c \in \mathbb{R}$ s.t. $f'(c) = 0$

contradiction.

Hence f is injective

so: $f : \mathbb{R} \to f(\mathbb{R})$ continuous bijection

• Let's prove that f^{-1} is continuous at $\underset{=f(b)}{a} \in f(\mathbb{R})$

Let $\varepsilon > 0$, $(f^{-1})^{-1}\left([b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]\right) = f\left([b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}]\right)$

is path-connected + compact since f is C^0,

$$f([b - \varepsilon/2, b + \varepsilon/2]) = [c, d]$$

assume $a = f(b) = c$ then b is a local extremum and $f'(b) = 0$

which is impossible, $\exists \delta > 0$ s.t.

$$(a - \delta, a + \delta) \subset (c, d) \subset f([b - \varepsilon/2, b + \varepsilon/2])$$

$$\Rightarrow f^{-1}((a - \delta, a + \delta)) \subset f^{-1} f([b - \varepsilon/2, b + \varepsilon/2]) \subset [b - \varepsilon/2, b + \varepsilon/2]$$

$b = f^{-1}(a)$

CCL: $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in f(\mathbb{R}), |x - x| < \delta \Rightarrow |f^{-1}(a) - f^{-1}(x)| \leq \varepsilon/2 < \varepsilon$

i.e. f^{-1} is C^0 at a

• $$\frac{f^{-1}(a) - f^{-1}(x)}{a - x} = \frac{f^{-1}(a) - f^{-1}(x)}{f(f^{-1}(a)) - f(f^{-1}(x))} \xrightarrow[x \to a]{} \frac{1}{f'(f^{-1}(a))}$$

So $(f^{-1})'(a) = 1/f'(f^{-1}(a))$

(2) • $\frac{f(x)-f(0)}{x-0} = \frac{x+x^2\sin(\pi/x)}{x} = 1 + x\sin(\pi/x) \xrightarrow[x\to 0]{} 1$

So $f'(0) = 1 \neq 0$

• Assume by contradiction that $\exists \varepsilon > 0$ s.t.

$f : (-\varepsilon,\varepsilon) \to f(-\varepsilon,\varepsilon)$ is a bijection

since f is continuous (since differentiable) and injective

on $(-\varepsilon,\varepsilon)$, it has to be increasing or decreasing on $(-\varepsilon,\varepsilon)$

but $f'(x) = 1 - \pi\cos(\pi/x) + 2x\sin(\pi/x)$ (for $x \in (-\varepsilon,\varepsilon)\setminus\{0\}$)

vanishes on $(-\varepsilon,\varepsilon)$ –

Contradiction.

(3) There is no contradiction

Here the function f is differentiable but not $\underline{\underline{C^1}}$

It means that we can't relax the C^1-assumption

in the Implicit Function Theorem

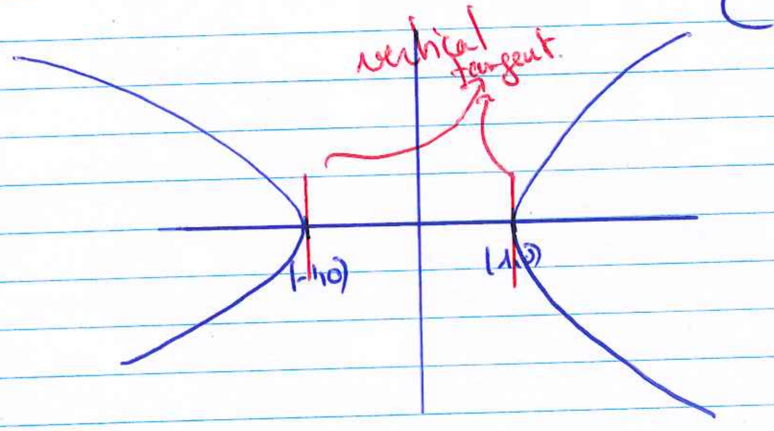
(2) addendum

• There is no contradiction with the Inverse FT since f is not C^1 (just differentiable)

• Here f' vanishes contrary to the function in (1)

Ex 7

(1)



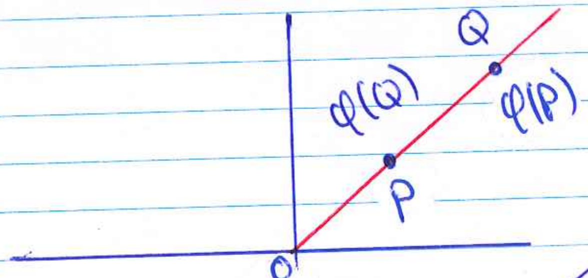
that's an hyperbola

(2) $\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is C^1 on $\mathbb{R}^2\setminus\{0\}$

$$\varphi(\varphi(v)) = \varphi\left(\frac{v}{\|v\|^2}\right) = \frac{\frac{v}{\|v\|^2}}{\left\|\frac{v}{\|v\|^2}\right\|^2} = \frac{\frac{v}{\|v\|^2}}{\frac{\|v\|^2}{\|v\|^4}} = v$$

So $\varphi^{-1} = \varphi : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$ which is C^1 also

hence $\varphi : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$ is a C^1-diffeomorphism



(φ maps v to a point in the same semiline s.t. $\overline{OP} \cdot \overline{O\varphi(P)} = 1$

when P is far/close to the origin then the image is close/far from the origin:
intuitively φ swaps "0" and "∞"

- $\|\varphi(v)\| \cdot \|v\| = \frac{\|v\|}{\|v\|^2} \cdot \|v\| = 1$

- $\|\varphi(v)\| = \frac{1}{\|v\|^2} \cdot v \leftarrow$ same semiline

③ $\varphi(x,y) = (X,Y) \Rightarrow (x,y) = \varphi^{-1}(X,Y) \overset{\uparrow}{=} \varphi(X,Y) = \left(\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2}\right)$

since $\varphi^{-1} = \varphi$

So $x = \frac{X}{X^2+Y^2}$, $\frac{Y}{X^2+Y^2}$

$(x-y)(x+y) = 1$

$\Leftrightarrow \left(\frac{X-Y}{X^2+Y^2}\right)\left(\frac{X+Y}{X^2+Y^2}\right) = 1$

$\Leftrightarrow \underbrace{(X-Y)(X+Y)}_{X^2-Y^2} = (X^2+Y^2)^2$

So an equation of $\varphi(C)$ is $\begin{cases} (X^2+Y^2)^2 = X^2-Y^2 \\ (X,Y) \neq (0,0) \end{cases}$

for the closure, we add $(0,0)$: $\tilde{C} = \overline{\varphi(C)} = \{(X^2+Y^2)^2 = X^2-Y^2\}$

$\tilde{C} : (X^2+Y^2)^2 = X^2 - Y^2$

④ $X = \rho\cos\theta$, $Y = \rho\sin\theta$

$\Rightarrow (\rho^2\cos^2\theta + \rho^2\sin^2\theta)^2 = \rho^2\cos^2\theta - \rho^2\sin^2\theta$

$\Rightarrow \rho^2 = \cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\Rightarrow \rho = \sqrt{\cos 2\theta}$ since $\rho \geq 0$

Hence $X = \cos\theta\sqrt{\cos 2\theta}$, $Y = \sin\theta\sqrt{\cos 2\theta}$

ie: A parametrization of $\tilde{C}$ is $\left[-\frac{\pi}{4},\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\frac{5\pi}{4}\right] \longrightarrow \mathbb{R}^2$

$\theta \longmapsto (\cos\theta\sqrt{\cos 2\theta}, \sin\theta\sqrt{\cos 2\theta})$

it is possible to modify this parametrization to obtain a better domain.

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}, \quad \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\cos(2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

hence $\quad X = \dfrac{\sqrt{1-\tan^2\theta}}{1+\tan^2\theta}, \quad Y = \dfrac{\tan\theta\sqrt{1-\tan^2\theta}}{1+\tan^2\theta}$

We define φ by $\cos\varphi = \tan\theta$, ie $\theta = \arctan(\cos\varphi)$

then $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \Leftrightarrow \varphi \in [-\pi, \pi]$

$$X = \frac{\sqrt{1-\cos^2\varphi}}{1+\cos^2\varphi} = \frac{\sin\varphi}{1+\cos^2\varphi}, \quad Y = \frac{\cos\varphi\sqrt{1-\cos^2\varphi}}{1+\cos^2\varphi} = \frac{\cos\varphi\sin\varphi}{1+\cos^2\varphi}$$

hence we have another paramtrization for $\tilde{c}$

$$\begin{aligned} [-\pi, \pi] &\longrightarrow \mathbb{R}^2 \\ \varphi &\longmapsto \left(\frac{\sin\varphi}{1+\cos^2\varphi}, \frac{\cos\varphi\sin\varphi}{1+\cos^2\varphi}\right) \end{aligned}$$

We can even go further by noticing that

$$\begin{aligned} (-\pi,\pi) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \tan(\varphi/2) \end{aligned}$$ is a bijection

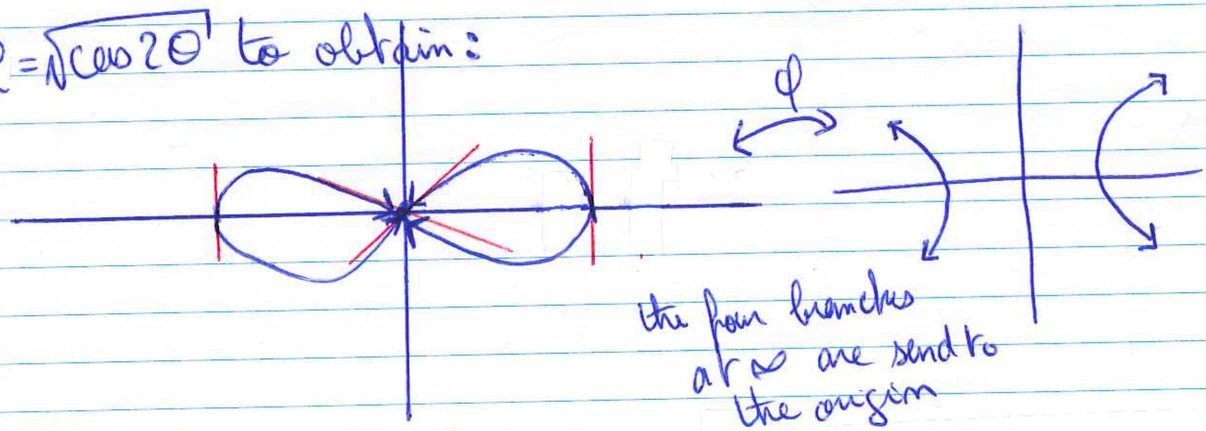
So we set $t = \tan(\varphi/2)$

$\sin\varphi = \frac{2t}{1+t^2}$, $\cos\varphi = \frac{1-t^2}{1+t^2}$, $1+\cos^2\varphi = 2\frac{1+t^4}{(1+t^2)^2}$

hence: $\tilde{C} = \sigma : \begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \left(\frac{t+t^3}{1+t^4}, \frac{t-t^3}{1+t^4}\right) \end{aligned}$

Rem $\lim_{t\to\pm\infty} \sigma(t) = (0,0) = \sigma(0)$ so we didn't miss anything by removing π and $-\pi$

(5) Use $\rho = \sqrt{\cos 2\theta}$ to obtain:



the four branches at ∞ are send to the origin

(6) Use $\sigma(t) = \left(\frac{t+t^3}{1+t^4}, \frac{t-t^3}{1+t^4}\right)$, $x'(t) = \frac{-t^6-3t^4+3t^2+1}{(1+t^4)^2}$

$= -\frac{(t-1)(t+1)(t^4+4t^2+1)}{(1+t^4)^2}$

$y'(1), y'(-1) \neq 0$

So $\forall t, \sigma'(t) \neq 0$ so the only issue is at 0

$t=0$: tangent: $\frac{y(t)}{x(t)} = \frac{t-t^3}{t+t^3} \longrightarrow 1$ so $\sigma(0)=0$ with tangent $y=x$

at ∞: tangent is $y=-x$ so self intersection

Exo8 :

① Let $t, s \in \mathbb{R}\setminus\{0\}$ $\quad (\sigma(t) = 0 \Leftrightarrow t = 0)$

Assume that $\sigma(t) = \sigma(s)$

ie $\begin{cases} \dfrac{t^3}{1+t^4} = \dfrac{s^3}{1+s^4} \\ \dfrac{t}{1+t^4} = \dfrac{s}{1+s^4} \end{cases}$

we divide the first line by line 2 (we can since $t, s \neq 0$) $\Rightarrow t^2 = s^2$

$\Rightarrow s = \pm t$

$s = -t$ in line 2

assume by contradiction that $s = -t$ then $\dfrac{t}{1+t^4} = \dfrac{-t}{1+t^4} \Rightarrow t = 0$

contradiction

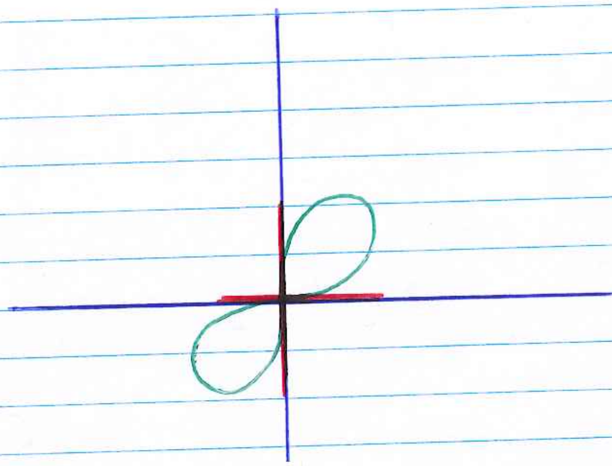
so $s = t$, ie σ is injective.

② $\sigma'(t) = \left(-\dfrac{t^2(t^4-3)}{(1+t^4)^2} , \dfrac{1-3t^4}{(1+t^4)^2} \right)$

$\left.\begin{array}{l} x'(t) \text{ vanishes for } t=0 \text{ or } t^4 = 3 \\ y'(t) \text{ vanishes for } t^4 = 1/3 \end{array}\right\} \forall t, \sigma'(t) \neq (0,0)$

③

t	$-\infty$		$-\sqrt[4]{3}$		$-\sqrt[4]{\frac{1}{3}}$		0		$\sqrt[4]{\frac{1}{3}}$		$\sqrt[4]{3}$		$+\infty$
$x'(t)$		$-$	|	$+$	|	$+$	0	$+$	|	$+$	|	$-$	
$x(t)$	0	↘	$\simeq -0{,}6$	↗	$\simeq -0{,}2$	↗	0	↗	$\simeq 0{,}2$	↗	$\simeq 0{,}6$	↘	0
$y(t)$	0	↘	$\simeq -0{,}3$	↘	$\simeq -0{,}6$	↗	0	↗	$\simeq 0{,}6$	↘	$\simeq 0{,}3$	↘	0
$y'(t)$		$-$	|	$-$	0	$+$	|	$+$	0	$-$	|	$-$	
$\frac{y(t)}{x(t)} = \frac{1}{t^2}$	0 horizontal tangent						$+\infty$ vertical						0 horizontal



When $t=0$, $\sigma(t)$ goes through $\sigma(0)=(0,0)$ with a vertical tangent line

When $t \to \pm\infty$ $\sigma(t)$ goes to $(0,0)$ with a horizontal tangent line

So there is a self intersection at $(0,0)$

ie the curve is singular at $(0,0)$

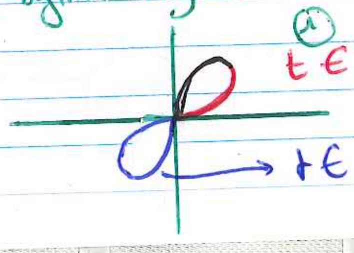
Comment : • $\sigma(-t) = -\sigma(t)$

So it is enough to study σ on $[0,+\infty)$ and then to do a symmetry w.r.t. the origin

• $\sigma(1/t) = \left(\frac{t^3}{1+t^4}, \frac{t}{1+t^4}\right)$: y and x are swapped

So it is enough to study σ on $[0,1]$ and then to do a symmetry w.r.t $y=x$

t	0		$\frac{1}{\sqrt[4]{3}}$		1
x'		$+$	0	$-$	
x	0	$\nearrow$		$\searrow$	$1/2$
y	0		$\nearrow$		$1/2$
y'	0		$+$		



① $t \in [0,1]$

② $t \in [1,+\infty]$ by symmetry w.r.t $y=x$

③ $t \in [-\infty,0]$ by symmetry w.r.t O

Ex 9

① $y^3 = x^2 \Leftrightarrow y = \sqrt[3]{x^2}$

but $x \mapsto (x, \sqrt[3]{x^2})$ is not C^1

notice that $\begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ t \mapsto t^3 \end{array}$ is a bijection

so we can set $x = t^3$

and then $y^3 = x^2 \Leftrightarrow \begin{cases} x = t^3 \\ y = t^2 \end{cases}$

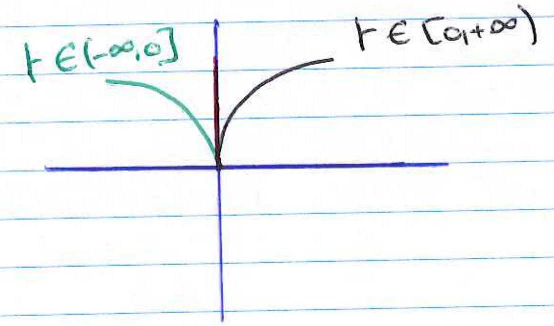
and $\sigma : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R}^2 \\ t \mapsto (t^3, t^2) \end{array}$ is C^1

Ccl $C = \{ (t^3, t^2) : t \in \mathbb{R} \}$

② $x(-t) = -x(t)$, $y(-t) = y(t)$

so it is enough to study σ on $[0, +\infty)$ and then to take the symmetry w.r.t. $x = 0$

t	0		∞
x'		$+$	
x	0	$\nearrow$	$+\infty$
y	0	$\nearrow$	$+\infty$
y'		$+$	



$\dfrac{y(t)}{x(t)} = \dfrac{1}{t}$, $\lim\limits_{t \to 0} \dfrac{y}{x} = +\infty$ so vertical tangent

③ Let $F(x,y) = y^3 - x^2$, $DF(x,y) = (-2x, 3y^2)$

rank $DF(x,y) < 1 \Leftrightarrow (x,y) = (0,0)$

So C is regular outside $(0,0)$

Let's prove by contradiction that σ is singular at $(0,0)$

- Assume that C is a C^1 graph $y=\varphi(x)$ around $(0,0)$

then $x^2 = y^3 \Rightarrow \varphi(x)^3 = x^2 \Rightarrow \varphi(x) = x^{2/3}$ not C^1 at 0

- Assume that C is a C^1 graph $x=\varphi(y)$ around $(0,0)$

then $\varphi(y)^2 = y^3 \Rightarrow \varphi(y) = \pm\, y^{3/2}$ not a graph

(for one y, we have 2 x-values in any neighborhood)

CCL: C is singular at $(0,0)$

Ex10: Let $F(x,y,z) = (x^2+y^2+z^2-1)^2$

$DF(x,y,z) = \left(4x(x^2+y^2+z^2-1) \quad 4y(x^2+y^2+z^2-1) \quad 4z(x^2+y^2+z^2-1)\right)$

DF is always $(0,0,0)$ on C so the theorem is not useful here. (algebraically, it sees a "self intersection" everywhere because of the square)

But Notice that $(x^2+y^2+z^2-1)^2 = 0$

$\Leftrightarrow x^2+y^2+z^2-1 = 0$

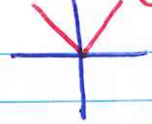
So $M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2-1 = 0\}$

Let $G(x,y,z) = x^2+y^2+z^2-1$

$DG(x,y,z) = (2x \quad 2y \quad 2z)$

So $\operatorname{rank} DG(x,y,z) < 1 = 3-2 \Leftrightarrow (x,y,z) = (0,0,0) \notin M$

So M is everywhere non-singular.

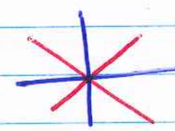
Ex 11 ① 

- assume that $\{(x,y) : y = \varphi(x),\ x \in (-\varepsilon,\varepsilon)\} = C_1 \cap B(0,r)$

then $y = |x| \Rightarrow \varphi(x) = |x|$ but $\varphi(x) = |x|$ is not C^1

- assume that $\{(x,y) : x = \varphi(y),\ y \in (-\varepsilon,\varepsilon)\} = C_1 \cap B(0,r)$

then $y = |x| \Rightarrow y = |\varphi(y)| \Rightarrow \varphi(y) = \pm y$ not a function
(in any neighborhood of 0, for one y values there are 2 possible x values)

② $x^2 = y^2 \Leftrightarrow x^2 - y^2 = 0 \Leftrightarrow (x-y)(x+y) = 0$ 

Method 1 : assume $\{y = \varphi(x),\ x \in (-\varepsilon,\varepsilon)\} = C_2 \cap B(0,r)$

then $(x - \varphi(x))(x + \varphi(x)) = 0 \Rightarrow \varphi(x) = \pm x$ two y values

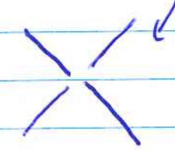
- assume $\{x = \varphi(y),\ y \in (-\varepsilon,\varepsilon)\} = C_2 \cap B(0,r)$

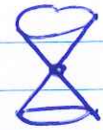
then again $\varphi(y) = \pm y$.
Contradiction

Method 2 : we know that if C_2 is regular at $(0,0)$ then $C_2 \cap B(0,r)$ is homeomorphic to a line $(a,b) \times \{0\}$

but $C_2 \cap B(0,r) \setminus \{(0,0)\}$ has 4 path-connected components

and $(a,b) \setminus \{pt\}$ has 2

Contradiction 

(3) C_3: 

<u>Method 1</u>:

- Assume that $C_3 \cap B(0,r) = \{x = \varphi(y,z)\}$

$$\varphi(y,z)^2 = z^2 - y^2 \Rightarrow \varphi(y,z) = \pm\sqrt{z^2-y^2} \rightsquigarrow \text{not a graph}$$

- $C_3 \cap B(0,r) = \{y = \varphi(x,z)\}$

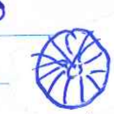
then $\varphi(x,z)^2 = z^2 - x^2 \Rightarrow \varphi(x,z) = \pm\sqrt{z^2-x^2} \rightsquigarrow$ not a graph

- $C_3 \cap B(0,r) = \{z = \varphi(x,y)\}$

then $\varphi(x,y)^2 = x^2+y^2 \Rightarrow \varphi(x,y) = \pm\sqrt{x^2+y^2} \rightsquigarrow$ not a graph

<u>Method 2</u>: We know that if C_3 is regular at $(0,0,0)$ then it is locally (<u>ie</u> $C_3 \cap B(0,r)$) homeomorphic to an <u>open</u> disk

But $C_3 \cap B(0,r) \setminus \{(0,0,0)\}$ =  has two path-connected comp

whereas $D \setminus \{pt\}$ has 1 :  Contradiction
↳disk

(4) $C_4 = \{x^3 = y^2\}$ 

Assume $C_4 \cap B(0,r) = \{y = \varphi(x) : x \in (-\varepsilon,\varepsilon)\}$

then $x^3 = y^2 \Rightarrow \varphi(x)^2 = x^3 \Rightarrow \varphi(x) = \pm x^{3/2}$ not a graph

Assume $C_4 \cap B(0,r) = \{x = \varphi(y) : y \in (-\varepsilon,\varepsilon)\}$

then $x^3 = y^2 \Rightarrow y^2 = \varphi(y)^3 \Rightarrow \varphi(y) = y^{2/3}$ not C^1

Ex 12 :

① $F(x,y,z) = \begin{pmatrix} x^2+y^2+z^2 & x^2+y^2-2x \end{pmatrix}$

$$DF(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ 2x-2 & 2y & 0 \end{pmatrix}$$

cofactors: $\begin{vmatrix} 2x & 2y \\ 2x-2 & 2y \end{vmatrix} = 4xy - 4y(x-1) = 4y$

$\begin{vmatrix} 2x & 2z \\ 2x-2 & 0 \end{vmatrix} = 4z(x-1)$

$\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix} = -4yz$

Case 1 : $y=0,\ z=0$

$\Rightarrow x^2 = R^2$ and $x^2 = 2x$

$\Rightarrow x = R = 2$ but $R \neq 2$

Case 2 : $x=1,\ y=0$ then $x^2+y^2-2x = -1 \neq 0$

so $(1,0,z) \notin C$

Hence C is not singular when $R \neq 2$

② By the above question, the only possible singular point when $R=2$ is $(2,0,0)$

$C \setminus \{(2,0,0)\}$ is regular of dim 1 (curve)

let's prove that $(2,0,0)$ is not regular of dim 1 in C

Assume $(x,y) = (x(z), y(z))$

or $(x,z) = (x(y), z(y))$

or $(y,z) = (y(x), z(x))$

and find a contradiction around $2,0,0$

Ex13 ① $\Rightarrow$ Assume that a is a regular point of dim d in M

then, up to permutations of the variables,

$$M \cap \mathcal{U} = \{(u, \varphi(u)),\ u \in B(\tilde{a}, \varepsilon)\},\ \tilde{a} = (a_1, \dots, a_d)$$

for some $\mathcal{U} \subset \mathbb{R}^N$ open and $\varphi : \underset{\mathbb{R}^d}{\underset{\cap}{B(\tilde{a}, \varepsilon)}} \longrightarrow \mathbb{R}^{N-d}$ C^1

Define $F : \mathcal{U} \longrightarrow \mathbb{R}^{N-d}$ by

$$F(\underbrace{x_1, \dots, x_d}_{u}, \underbrace{x_{d+1}, \dots, x_N}_{v}) = \varphi(x_1, \dots, x_d) - (x_{d+1}, \dots, x_N) = \varphi(u) - v$$

then F is C^1 and

$$\mathcal{U} \cap M = \{(u, \varphi(u)),\ u \in B(\tilde{a}, \varepsilon)\}$$
$$= \{(u, v) \in \mathcal{U},\ v = \varphi(u)\}$$
$$= \{(u, v) \in \mathcal{U},\ F(u, v) = 0\}$$

$\Leftarrow$: Assume that $F : \underset{\mathbb{R}^N}{\overset{\cup}{\mathcal{U}}} \to \mathbb{R}^{N-d}$ is C^1, $a \in \mathcal{U}$, $\mathrm{rank}(DF(a)) = N-d$

and $\mathcal{U} \cap M = F^{-1}(\vec{0})$.

By the implicit function theorem, locally around a, we may express $N-d$ components as a C^1 function of d components

Indeed, $\exists\, x_{i_1}, \dots, x_{i_{N-d}}$ variables s.t.

$D_{x_{i_1}, \dots, x_{i_{N-d}}} F(a)$ is invertible by the assumption

(2) (a) $T_aM \subset \ker DF(a)$:

Let $v \in T_aM$, then $\exists \gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}^N$ C^1 s.t.
$\gamma(0) = a$, $\forall t \in (-\varepsilon,\varepsilon)$, $\gamma(t) \in M$ and $v = \gamma'(0)$

Then $F(\gamma(t)) = 0$ $\forall t$ since $\gamma(t) \in M \cap \mathcal{U}$ for t close enough to 0

$\Rightarrow$ $DF(\gamma(0)) \cdot \gamma'(0) = 0$ by the chain rule

$\Rightarrow$ $DF(a) \cdot v = 0$

ie $v \in \ker DF(a)$

$\ker DF(a) \subset T_aM$:

Let $v \in \ker DF(a)$. Set $\tilde{a} = (a_1, \dots, a_d)$ where $a = (a_1, \dots, a_N)$

By the implicit function theorem (up to a permutation of the variables), there exists $\varphi : B(\tilde{a}, \varepsilon) \to \mathbb{R}^{N-d}$ s.t. around a

$$F(x,y) = 0 \iff y = \varphi(x) \qquad x = (x_1, \dots, x_d), \quad y = (x_{d+1}, \dots, x_N)$$

Define $\gamma(t) = (a_1 + tv_1, \dots, a_d + tv_d, \varphi(a_1 + tv_1, \dots, a_d + tv_d))$

then (1) γ is C^1 and $\forall t$, $F(\gamma(t)) = 0$ ie $\gamma(t) \in M$

(2) $\gamma(0) = (a_1, \dots, a_d, \varphi(a_1, \dots, a_d)) = (a_1, \dots, a_N) = a$

(3) Since $v \in \ker DF(a)$:

$$0 = DF(a)v = (D_xF(a) \;\; D_yF(a)) \begin{pmatrix} v_1 \\ \vdots \\ v_d \\ v_{d+1} \\ \vdots \\ v_N \end{pmatrix} \qquad r = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \quad s = \begin{pmatrix} v_{d+1} \\ \vdots \\ v_N \end{pmatrix}$$

$$= D_xF(a)r + D_yF(a)s$$

$$\Rightarrow s = -[D_yF(a)]^{-1} D_xF(a) r$$

then $\gamma'(0) = \begin{pmatrix} v_1 \\ \vdots \\ v_d \\ D\varphi(a)\begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} r \\ -[D_yF(a)]^{-1} D_xF(a) r \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} = v$

So $v \in T_aM$

(b) By the rank-kernel theorem

$$\dim(\mathbb{R}^N) = \dim\ker DF(a) + \operatorname{rank} DF(a)$$

ie $\quad N = \dim\ker DF(a) + N-d$

$\Rightarrow \dim\ker DF(a) = d$

So $T_aM = \ker DF(a)$ is a d-dim vector space

(c) $U\cap M = \{(x,y) : x\in B(\tilde{a},\varepsilon),\ y=\varphi(x)\}$, $\tilde{a} = (a_1,\dots,a_d)$

Let $v\in T_aM$, ie $v=\gamma'(0)$ for γ as in (2)

Since $\forall t,\ \gamma(t)\in M$, $(\gamma_{d+1}(t),\dots,\gamma_N(t)) = \varphi(\gamma_1(t),\dots,\gamma_d(t))$

$$\Rightarrow (\gamma_{d+1}'(0),\dots,\gamma_N'(0)) = D\varphi(\gamma_1(0),\dots,\gamma_d(0))\cdot\begin{pmatrix}\gamma_1'(0)\\ \vdots\\ \gamma_d'(0)\end{pmatrix}$$

ie $(v_{d+1},\dots,v_N) = D\varphi(\tilde{a})\begin{pmatrix}v_1\\ \vdots\\ v_d\end{pmatrix}$

So $v\in$ Graph of $D\varphi(\tilde{a}) = \Gamma_{D\varphi(\tilde{a})}$

So $T_aM\subset\Gamma_{D\varphi(\tilde{a})}$ and they are both vector subspace of dim d

Hence $T_aM = \Gamma_{D\varphi(\tilde{a})}$

Ex 14 :

① • f is C^1, $Df(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ same dim

$$\det Df(x,y) = e^{2x} \neq 0$$

So the Inverse function theorem apply everywhere

ie : $\forall (x_0,y_0) \in \mathbb{R}^2$, $\exists\, \mathcal{U}, \mathcal{V} \subset \mathbb{R}^2$ open s.t.

$$\begin{array}{ccc} f : \mathcal{U} & \longrightarrow & \mathcal{V} \quad C^1 \text{ diffeo} \\ \Psi & & \Psi \\ (x_0,y_0) & & f(x_0,y_0) \end{array}$$

• $\triangle$ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not a C^1 diffeo

$f(0,0) = f(0, 2\pi)$ so f is not injective

② Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x,y) = y$,

then $\mathcal{U} = \varphi^{-1}((0,2\pi))$ is open as the inverse image of an open set by a continuous function.

③ Let's prove that f is injective on $\mathcal{U}$

Let $(x_0,y_0), (x_1,y_1) \in \mathcal{U}$ s.t. $f(x_0,y_0) = f(x_1,y_1)$

$$\begin{cases} e^{x_0} \cos y_0 = e^{x_1} \cos(y_1) \\ e^{x_0} \sin(y_0) = e^{x_0} \sin(y_1) \end{cases} \Rightarrow e^{x_0} = e^{x_1} \quad \left(\begin{array}{c}\text{square} \\ \text{and sum}\end{array}\right)$$

$\Rightarrow x_0 = x_1$ since e is injective

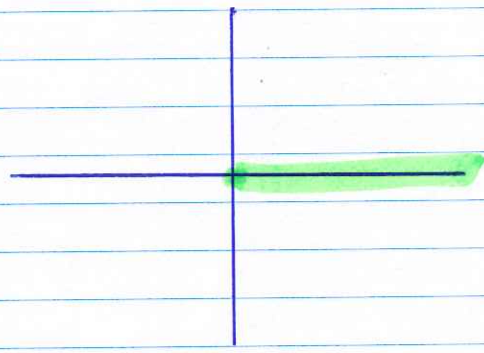
then $\begin{cases} \cos(y_0) = \cos(y_1) \\ \sin(y_0) = \sin(y_1) \end{cases} \Rightarrow y_0 = y_1$ since $y_i \in (0, 2\pi)$

So f is injective on $\mathcal{U}$

f injective on U + $\forall (x,y) \in U$, $Df(x,y)$ invertible

$\Rightarrow f(U)$ is open and $f : U \to f(U)$ is a C^1-diffeo

Rem: $f(U) = \mathbb{R}^2 \setminus \{(x,0), x \geq 0\}$



take $(x,y) \in \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$

then $(x,y) = r(\cos\theta, \sin\theta)$, $\theta \in (0, 2\pi)$

Since $r > 0$, $\exists \rho$ s.t. $r = e^{\rho}$

hence $(x,y) = f(\rho, \theta)$

and $(x,y) \in f(U)$

ie $\mathbb{R}^2 \setminus \{(x,0), x \geq 0\} \subset f(U)$

the other direction is easy.

(4) $f \circ g = id \xRightarrow{\text{chain rule}} Df(g(0,1))\, Dg(0,1) = I_{2,2}$

$\Rightarrow Df(0, \frac{\pi}{2})\, Dg(0,1) = I_{2,2}$

Since $f(0, \frac{\pi}{2}) = (0,1)$

$\Rightarrow Dg(0,1) = \left[Df(0, \frac{\pi}{2})\right]^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Ex 15

① Let $\varepsilon > 0$

Since f is integrable, there exists a partition P s.t.

$$U_P(f) - L_P(f) < \varepsilon$$

ie $\sum_S \vartheta(S)\left(\sup_S(f) - \inf_S(f)\right) < \varepsilon$ where S goes through the subrectangles of P (finitely many!)

Notice that the graph of f satisfies

$$\Gamma_f \subset \bigcup_S \left(S \times [\inf_S f, \sup_S f]\right)$$

$$\cap$$

$$\{(x, f(x)) : x \in S\}$$

but $\sum_S \vartheta\left([\inf_S f, \sup_S f] \times S\right) = \sum_S \vartheta(S)\left(\sup_S f - \inf_S f\right) < \varepsilon$

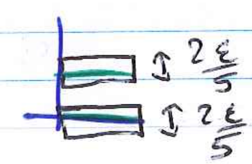
So Γ_f has zero content

② the converse is false

$f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

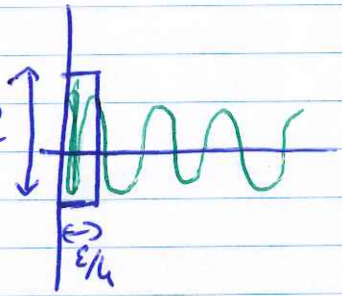
Then f is NOT integrable but

$$\Gamma_f \subset \left([0,1] \times [-\tfrac{\varepsilon}{5}, \tfrac{\varepsilon}{5}]\right) \cup \left([0,1] \times [1-\tfrac{\varepsilon}{5}, 1+\tfrac{\varepsilon}{5}]\right)$$



of area $\frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} = \frac{4\varepsilon}{5} < \varepsilon$

Ex 16



Let $\varepsilon > 0$.

Notice that $f : [\frac{\varepsilon}{4}, 5] \to \mathbb{R}$ is continuous hence integrable hence its graph has zero content by the previous question: it is covered by $R_1, \dots, R_q$ rectangles s.t. $\partial(R_1) + \dots + \partial(R_q) < \varepsilon/2$

then the graph of f on $(0,5]$ is covered by $R_0, R_1, \dots, R_q$ where $R_0 = [0, \varepsilon/4] \times [-1, 1]$

and $\partial(R_0) + \partial(R_1) + \dots + \partial(R_q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Ex17 :

① $\mathbb{N}$ is not bounded

② Assume that $[0,1]$ has zero content by contradiction

then $[0,1] \subset \bigcup_{i=1}^{r} [a_i, b_i]$ with $\sum b_i - a_i < 1/2$

$\Rightarrow \underbrace{\vartheta([0,1])}_{=1} \leq \sum \vartheta([a_i, b_i]) = \sum b_i - a_i < 1/2$

Contradiction

③ $\mathbb{Q} \cap [0,1]$ has zero content $\Leftrightarrow$ $\underbrace{\overline{\mathbb{Q} \cap [0,1]}}_{=[0,1]}$ has zero content

④ let $\varepsilon > 0$

$[0,1] \times \{0\} \subset [0,1] \times [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$

$\vartheta([0,1] \times [-\varepsilon/4, \varepsilon/4]) = \frac{\varepsilon}{2} < \varepsilon$

Ex18 : $\{x^2 + y^2 = 1\} = \{x^2+y^2 = 1, y \geq 0\} \cup \{x^2+y^2 = 1, y \leq 0\}$

$=$ graph of $(\sqrt{1-x^2})$ $\cup$ graph of $(-\sqrt{1-x^2})$

Ex 19: Let $\varepsilon > 0$

$$\frac{1}{m} \xrightarrow[m \to \infty]{} 0$$

So $\exists N \in \mathbb{N}$ st. $m \geq N \Rightarrow \left| \frac{1}{m} - 0 \right| < \varepsilon/2$

ie $\frac{1}{m} \in [0, \varepsilon/2]$

$A = \left\{ \frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{N-1} \right\}$ is finite and hence has zero content

ie $[a_1, b_1], \dots, [a_m, b_m]$ s.t.

$$\sum_{i=1}^{m} (b_i - a_i) < \varepsilon/2$$

and $A \subset \bigcup_{i=1}^{m} [a_i, b_i]$

Hence $\left\{ \frac{1}{m}, m \in \mathbb{N}_{>0} \right\} = A \cup \left\{ \frac{1}{m}, m \geq N \right\}$

$$\subset \left(\bigcup_{i=1}^{m} [a_i, b_i] \right) \cup [0, \varepsilon/2]$$

and $\sum (b_i - a_i) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$

So $\left\{ \frac{1}{m}, m \in \mathbb{N}_{>0} \right\}$ has zero content

Remark : $p \wedge q = \gcd(p,q)$

Exo 20

① • Let $x_0 \in [0,1] \cap \mathbb{Q}$

We want to show that f is not C^0 at x_0

ie $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in [0,1], |x - x_0| < \delta$ and $|f(x) - f(x_0)| \geqslant \varepsilon$

Let denote $x_0 = p/q$, $p \wedge q = 1$, $q \in \mathbb{N}_{>0}$

then $f(x_0) = 1/q$

Define $x_i = x_0 + \frac{\sqrt{2}}{i}$ for $i \in \mathbb{N}_{>0}$ (or $x_i = x_0 - \frac{\sqrt{2}}{i}$ if $x_0 = 1$)

For i large enough, $x_i \in [0,1]$ and $x_i \notin \mathbb{Q}$ so $f(x_i) = 0$

Let $\delta > 0$. Since $x_i \xrightarrow[i \to +\infty]{} x_0$, $\exists i$ large enough

so that $x_i \in [0,1]$ and $|x_0 - x_i| < \delta$

but $|f(x_0) - f(x_i)| = \frac{1}{q}$ (= our ε)

• Let $x_0 \in [0,1] \setminus \mathbb{Q} = (0,1) \setminus \mathbb{Q}$

We want to show that f is C^0 at x_0

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$ (since $\frac{1}{n} \xrightarrow[n \to \infty]{} 0$) ↗ $\mathbb{R}$ is archimedean

We can prove that $(0,1)$ contains finitely many rationals

of the form p/q, $p \wedge q = 1$, $q \in \mathbb{N}_{>0}$, $q \leq N$ (Next page)

So it is possible, since $x_0 \notin \mathbb{Q}$, to find $\delta > 0$ s.t.

$(x_0 - \delta, x_0 + \delta) \subset (0,1)$ doesn't contain any such rationals

So if $x \in (x_0 - \delta, x_0 + \delta)$ $\begin{cases} x \in \mathbb{Q} \Rightarrow |f(x) - f(x_0)| = \frac{1}{q} < \frac{1}{N} < \varepsilon \\ x \notin \mathbb{Q} \Rightarrow |f(x) - f(x_0)| = 0 < \varepsilon \end{cases}$

Lemma: $I=(a,b)\subset(0,+\infty)$, $a,b\in\mathbb{R}_{>0}$, $N\in\mathbb{N}$

The set $\{p/q\in(a,b) : p\in\mathbb{Z}, q\in\mathbb{N}_{>0}, p\wedge q=1, q\leq N\}$ is finite

Δ Let p/q be in the above set then

$$0<a<p/q<b$$

$$\Rightarrow 0<aq<p<bq\leq bN$$

but $p\in\mathbb{Z}$ so there are finitely many possible p

and $q\leq N$ so there are finitely many possible q $\square$

(2) • For any partition, $[x_k,x_{k+1}]\cap\mathbb{Q}\neq\emptyset$

so $L_P(f)=0$

• Given $\varepsilon>0$, we may use the above lemma to obtain that there are finitely many rationals in $[0,1]$ s.t. $f(x)>\varepsilon/2$

we put them in intervals of lengths $\frac{\varepsilon}{2N}$ where $N=\#$ of such intervals

then $$U_P(f)=\sum_{k=0}^{n-1}(x_{k+1}-x_k)\sup_{[x_k,x_{k+1}]}f$$

$$=\sum_{\substack{\exists x\in[x_k,x_{k+1}]\\ f(x)>\varepsilon/2}}\overbrace{(x_{k+1}-x_k)}^{<\varepsilon/2N}\overbrace{\sup f}^{\leq 1}+\sum_{\text{otherwise}}(x_{k+1}-x_k)\sup f$$

$$<\varepsilon/2+\overbrace{\sum(x_{k+1}-x_k)}^{\leq 1}\frac{\varepsilon}{2}$$

$$\leq\varepsilon$$

So $U_P-L_P<\varepsilon$

③ $[0,1] \cap \mathbb{Q}$ has zero content

$$\Leftrightarrow \overline{[0,1] \cap \mathbb{Q}} \text{ has zero content}$$

$$\overline{[0,1] \cap \mathbb{Q}} = [0,1]$$

the latter is false

④ f integrable $\not\Rightarrow$ the discontinuity set has zero content

(Thomae's function is a counter-example)