University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

Jean-Baptiste Campesato

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Exercise 1. *Prove that f admits a maximum and a minimum on K and find them for the following data:*

1. $f(x,y) = xy(1-x-y)$, $K = \left\{(x,y) \in \mathbb{R}^2 \ :\ x \geq 0,\ y \geq 0,\ x+y \leq 1\right\}$
2. $f(x,y) = y^2 - x^2y + x^2$, $K = \left\{(x,y) \in \mathbb{R}^2 \ :\ x^2 - 1 \leq y \leq 1 - x^2\right\}$
3. $f(x,y,z) = x + 2y + 3z$, $K = \left\{(x,y,z) \in \mathbb{R}^3 \ :\ x^2 + y^2 = 1,\ y - z = 2\right\}$

Exercise 2. *Show that the point of the line $2x - y = 16$ which is the closest to the origin is well-defined and find it.*

Exercise 3. *Prove that the level set $x^4 + x^3y^2 - y + y^2 + y^3 = 1$ defines a C^1 function $y = \varphi(x)$ in a neighborhood of $(-1,1)$. Compute $\frac{\partial \varphi}{\partial x}(-1)$.*

Exercise 4. *Prove that the level set $x + y + z + sin(xyz) = 0$ defines a C^1 function $y = \varphi(x,z)$ in a neighborhood of $(0,0,0)$. Compute $\frac{\partial \varphi}{\partial x}(0,0)$ and $\frac{\partial \varphi}{\partial z}(0,0)$. Compute $D\varphi(0)$.*

Exercise 5. *Prove that the level set $\left\{(x,y,z) \in \mathbb{R}^3 \ :\ x^2 - y^2 + 2z^2 = 2,\ xyz = 1\right\}$ defines a C^1 function $(x,z) = \varphi(y)$ in a neighborhood of $(1,1,1)$ (i.e. $x = \varphi_1(y)$ and $z = \varphi_2(y)$).*
Compute $\frac{\partial \varphi_1}{\partial y}(1)$ and $\frac{\partial \varphi_2}{\partial y}(1)$. Compute $D\varphi(1)$.

Exercise 6 (Difficult)**.**

1. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that if $\forall x \in \mathbb{R},\ f'(x) \neq 0$ then $f : \mathbb{R} \to f(\mathbb{R})$ is a homeomorphism and $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$ is differentiable..*
2. *Define $f : \mathbb{R} \to \mathbb{R}$ by* $f(x) = \begin{cases} x + x^2 \sin(\pi/x) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$.
   Prove that $f'(0)$ exists and is not 0 but that f is not invertible in any neighborhood of the origin.
   What's the difference with the previous question? Is there any contradiction with the inverse mapping theorem?
3. *Define $F : \mathbb{R}^2 \to \mathbb{R}$ by* $F(x,y) = \begin{cases} \frac{y^3 - x^8 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$.
   We may prove (you don't have to do it) that F is every differentiable, that $\frac{\partial F}{\partial y}(0,0) \neq 0$ but that the level set $\left\{(x,y) \in \mathbb{R}^2 \ :\ F(x,y) = F(0,0)\right\}$ can't be described as the graph of a C^1-function $y = \varphi(x)$ locally around $(0,0)$.
   Is there any contradiction with the implicit function theorem?

Exercise 7 (Bernoulli's lemniscate, very difficult)**.** *We define $C = \left\{(x,y) \in \mathbb{R}^2 \ :\ (x-y)(x+y) = 1\right\}$.*

1. *Draw C.*
2. *Prove that $\varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ defined by $\varphi(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|^2}$ is a C^1-diffeomorphism. Find φ^{-1}.*
   What is the geometric interpretation of φ?
3. *Find an equation for $\tilde{C} = \overline{\varphi(C)}$. Intuitively, how do we pass from C to $\tilde{C}$?*
4. *Using polar coordinates, find a C^1 parametrization of $\tilde{C}$.*
5. *Sketch $\tilde{C}$.*
6. *Study the singular points of $\tilde{C}$.*

Exercise 8 (Another lemniscate)**.** *Define $\sigma : \mathbb{R} \to \mathbb{R}^2$ by* $\sigma(t) = \left(\frac{t^3}{1+t^4}, \frac{t}{1+t^4}\right)$.

1. Prove that σ is injective. *2. Prove that $\forall t \in \mathbb{R},\ \sigma'(t) \neq \mathbf{0}$.* *3. Is $C = \{\sigma(t) \ :\ t \in \mathbb{R}\}$ regular?*

Exercise 9 (Ordinary cusp)**.** *Let $C = \left\{(x,y) \in \mathbb{R}^2 \ :\ x^2 - y^3 = 0\right\}$.*

1. Find a C^1 parametrization of C. *2. Sketch C.* *3. Study the singularities of C.*

Exercise 10. *Study the singular points of* $M = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2 + z^2 - 1)^2 = 0\right\}$

Exercise 11. *Prove that the following sets are singular at the origin.*
1. $C_1 = \left\{(x, y) \in \mathbb{R}^2 \ : \ y = |x|\right\}$ 2. $C_2 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 = y^2\right\}$
3. $C_3 = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2\right\}$ 4. $C_4 = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^3 = y^2\right\}$

Exercise 12. *Let* $C = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = R^2,\ x^2 + y^2 - 2x = 0\right\}$ *where* $R > 0$.
1. *Prove that* C *is non-singular for* $R \neq 2$. 2. *Study* C *when* $R = 2$.

Exercise 13. *Let* $M \subset \mathbb{R}^N$ *and* $\mathbf{a} \in M$.

1. *Prove that* $\mathbf{a}$ *is a regular point of dimension* d *of* M *if and only if there exists* $U \subset \mathbb{R}^N$ *an open subset containing* $\mathbf{a}$ *and* $\mathbf{F} : U \to \mathbb{R}^{N-d}$ *a* C^1 *function such that* $\operatorname{rank}(D\mathbf{F}(a)) = N - d$ *and* $U \cap M = \{\mathbf{x} \in U \ : \ \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$.
2. *Now, we assume that* a *is regular of dimension* d *and we define the tangent space of* M *at* a *by* $T_{\mathbf{a}}M = \left\{\gamma'(0) \,\middle|\, \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^N\ C^1,\ \forall t \in (-\varepsilon, \varepsilon),\ \gamma(t) \in M,\ \gamma(0) = \mathbf{a}\right\}$.
   (*a*) *Let* $\mathbf{F}$ *be as in the previous question. Prove that* $T_{\mathbf{a}}M = \ker D\mathbf{F}(\mathbf{a})$.
   (*b*) *Deduce that* $T_{\mathbf{a}}M$ *is a vector space of dimension* d.
   (*c*) *Assume that* M *is locally a graph* $(x_{d+1}, \dots, x_N) = \varphi(x_1, \dots, x_d)$ *around* $\mathbf{a}$.
   Prove that $T_{\mathbf{a}}M$ *is the graph of* $D\varphi(a_1, \dots, a_d)$.

Exercise 14. *Let* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $\mathbf{f}(x, y) = (e^x \cos y, e^x \sin y)$.
1. *For which points in* $\mathbb{R}^2$ *are the assumptions of the inverse function theorem satisfied? What can we conclude? Is* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{f}(\mathbb{R}^2)$ *a global* C^1*-diffeomorphism?*
2. *Prove that* $U = \left\{(x, y) \in \mathbb{R}^2 \ : \ y \in (0, 2\pi)\right\}$ *is open.*
3. *Prove that* $\mathbf{f}(U)$ *is open and that* $\mathbf{f} : U \to \mathbf{f}(U)$ *is a* C^1*-diffeomorphism. What is* $f(U)$*?*
4. *If we denote* $\mathbf{g} = \mathbf{f}^{-1} : f(U) \to \mathbf{U}$*, compute* $D\mathbf{g}(0, 1)$.

Exercise 15. *Let* $R \subset \mathbb{R}^n$ *be a rectangle (or segment line for* $n = 1$*) and* $f : R \to \mathbb{R}$.
1. *Prove that if* f *is integrable then its graph* $\Gamma_f = \{(x, f(x)) \ : \ x \in R\} \subset \mathbb{R}^{n+1}$ *has zero content.*
2. *Prove that the converse is false.*

Exercise 16. *Prove that* $\left\{\left(x, \sin\frac{1}{x}\right) \ : \ x \in (0, 5]\right\}$ *has zero content.*

Exercise 17.
1. *Prove that* $\mathbb{N}$ *doesn't have zero content.*
2. *Prove that* $[0, 1]$ *doesn't have zero content.*
3. *Prove that* $\mathbb{Q} \cap [0, 1]$ *doesn't have zero content.*
4. *Prove that* $[0, 1] \times \{0\}$ *has zero content.*

Exercise 18. *Prove that* $\left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\right\}$ *has zero content.*

Exercise 19. *Prove that* $\left\{\frac{1}{n} \ : \ n \in \mathbb{N}_{>0}\right\}$ *has zero content (even if it is not finite).*

Exercise 20 (Thomae's function, difficult). *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ p \in \mathbb{Z} \setminus \{0\},\ q \in \mathbb{N}_{>0},\ \gcd(p, q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

1. *Prove that* f *is discontinuous at all the rational points but is continuous at all irrational numbers.*
2. *Prove that* f *is integrable on* $[0, 1]$ *using the definition of integrability.*
3. *Prove that* $[0, 1] \cap \mathbb{Q}$ *doesn't have zero content (hint: compute* $\overline{[0, 1] \cap \mathbb{Q}}$*).*
4. *We know that if the discontinuity set of a function* $f : [a, b] \to \mathbb{R}$ *has zero content then* f *is integrable. Is the converse true?*

Exercise 21 (Dirichlet's function). *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

1. *Prove that* f *is nowhere continuous.*
2. *Prove that* f *is not integrable on* $[0, 1]$ *using the definition of integrability.*

Exercise 22 (Uniform continuity).
Solve the questions from §B of `http://www.math.toronto.edu/campesat/ens/1920/darboux.pdf`
You'll find the solutions at `http://www.math.toronto.edu/campesat/ens/1920/0121-notes.pdf`
Beware: Uniform continuity is more subtle than what it looks like at first glance...
I think it is very important to practice these questions to develop your intuition about uniform continuity.