# Infimum and Supremum (Recollection from MAT137)

**Def:** $A \subset \mathbb{R}$, $L, U \in \mathbb{R}$.
We say that:

- $L$ is a lower bound of $A$ if $\forall x \in A,\ L \leq x$
- $U$ is an upper bound of $A$ if $\forall x \in A,\ x \leq U$

**Def:** $A \subset \mathbb{R}$, $S \in \mathbb{R}$
We say that $S$ is the supremum (or least upper bound) of $A$ if

① $S$ is an upper bound of $A$
ie: $\forall x \in A,\ x \leq S$

② it is the least one
ie: $T$ is an upper bound of $A \Rightarrow S \leq T$

**Def:** $A \subset \mathbb{R}$, $I \in \mathbb{R}$
We say that $I$ is the infimum (or greatest lower bound) of $A$ if

① $I$ is a lower bound of $A$
ie: $\forall x \in A,\ I \leq x$

② it is the greatest one
ie: $J$ is a lower bound of $A \Rightarrow J \leq I$

Fundamental property of $\mathbb{R}$: "Dedekind completeness"

⚠ LUB principle: a non-empty subset of $\mathbb{R}$ which is bounded from above admits a supremum

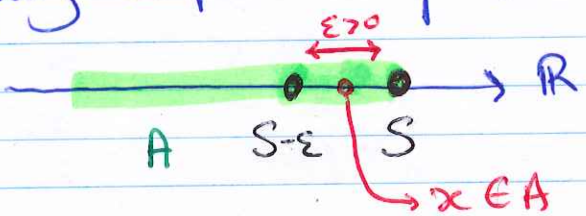
⚠ GLB principle: a non-empty subset of $\mathbb{R}$ which is bounded from below admits an infimum

<u>Theorem</u>: $A \subset \mathbb{R}$, $S, I \in \mathbb{R}$

$$S = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A, \ x \leq S \\ \forall \varepsilon > 0, \ \exists x \in A, \ S - \varepsilon < x \end{cases}$$

$$I = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A, \ I \leq x \\ \forall \varepsilon > 0, \ \exists x \in A, \ x < I + \varepsilon \end{cases}$$

$\Delta$ I am only going to prove the first one:




$\Rightarrow$: By definition, $\forall x \in A, x \leq S$

Let $\varepsilon > 0$, then $S - \varepsilon < S$ and hence $S - \varepsilon$ is not an upper bound of $A$ since $S$ is the least one.

<u>ie</u>: $\exists x \in A, \ S - \varepsilon < x$

<u>$\Leftarrow$</u>: $\forall x \in A, \ x \leq S$ means that $S$ is an upper bound of $A$

<u>Let's prove</u> it is the least one.

We will prove the contrapositive: $T < S \Rightarrow T$ is not an upper bound

Let $T \in \mathbb{R}$. Assume that $T < S$.

Set $\varepsilon = S - T$, then $\varepsilon > 0$.

By assumption, $\exists x \in A$ s.t. $S - \varepsilon < x$ <u>ie</u> $T < x$

So $T$ is not an upperbound

$\square$

## Integration

**Comment 1:** Be sure that you remember the one-variable construction from last year

(There is a recap on my webpage)

**Comment 2:** We are going to talk about the integral of Darboux (but with several variables)

Historical comment, you can skip it

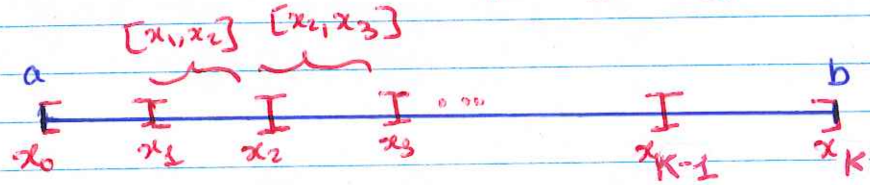
- Since over $\mathbb{R}$ this construction gives the formerly defined Riemann's integral, it is common to simply call the result of both definitions "Riemann's integral"

- $\left.\begin{array}{l}\text{Darboux integrable} \Rightarrow \text{Riemann integrable} \\ \text{Archimedean property}\end{array}\right\}$ is another characterization of the Dedekind-completeness of $\mathbb{R}$

**Definition:** A *partition* of the segment line $[a,b]$ is a finite subset of $[a,b]$ containing $a$ and $b$.

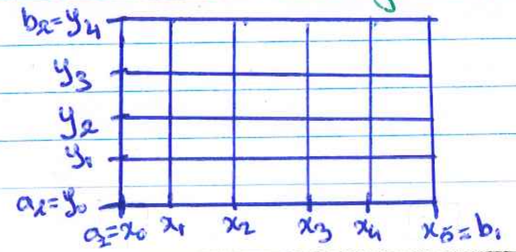
i.e. $P = \{a = x_0 < x_1 < \dots < x_K = b\}$

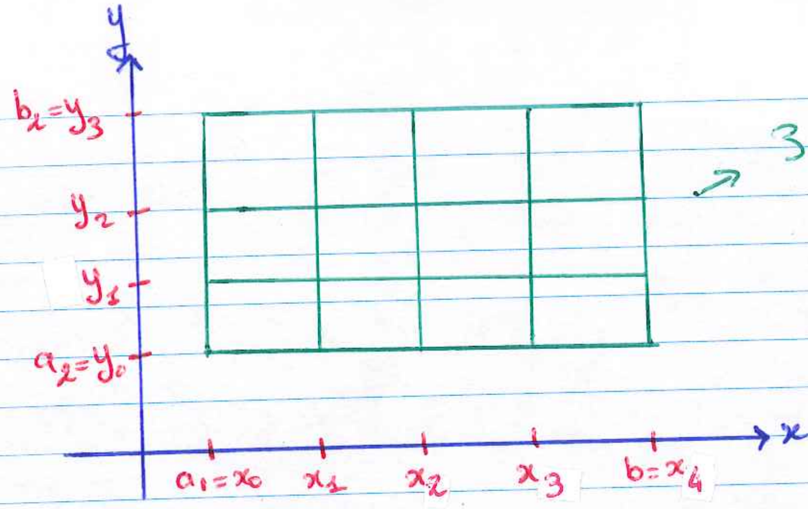
Intuitively, we break $[a,b]$ into finitely many closed subintervals.




**Definition:** A *partition* of the rectangle $[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_m,b_m]$ is a collection $P=(P_1,\dots,P_m)$ where $P_i$ is a partition of $[a_i,b_i]$

Intuitively, we break the rectangle into finitely many closed subrectangles






$\nearrow$ $3 \times 4 = 12$ subrectangles

<u>Remark</u>: • a partition $P = \{a = x_0 < x_1 < \dots < x_K = b\}$ contains $K$ subintervals

• a partition $P = (P_1, \dots, P_m)$ of $[a_1, b_1] \times \dots \times [a_m, b_m]$ contains $K_1 \cdot K_2 \cdots K_m$ subrectangles where

$$P_i = \{a_i = x_{i0} < x_{i1} < \dots < x_{iK_i} = b_i\}$$

<u>Definition</u>: the **length** of an interval $[a, b]$ is $\mathcal{V}([a,b]) := b - a$

<u>Definition</u>: the **volume** of a rectangle $S = [a_1, b_1] \times \dots \times [a_m, b_m]$ is

$$\mathcal{V}(S) := (b_1 - a_1) \cdot (b_2 - a_2) \cdots (b_m - a_m)$$
$$= \mathcal{V}([a_1, b_1]) \, \mathcal{V}([a_2, b_2]) \cdots \mathcal{V}([a_m, b_m])$$

<u>Definition</u>: Let $P, Q$ be two partitions of $[a, b]$.
We say that $Q$ is **finer** than $P$ if $P \subset Q$

$\left\{ \text{I recall that } P \subset Q \text{ means } P \subseteq Q \text{ (subset or equal)} \right.$


<u>Definition</u>: Let $P = (P_1, \dots, P_m)$, $Q = (Q_1, \dots, Q_m)$ be 2 partitions of $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_m, b_m]$.

We say that $Q$ is **finer** than $P$ if $\forall i$, $P_i \subset Q_i$

<u>Remark</u>: Given two partitions $P$ and $Q$, it is always possible to find a third partition $O$ which is finer than both $P$ and $Q$

Let $O = P \cup Q$ (or $O_i = P_i \cup Q_i$ for a rectangle)

<u>Definition</u>: Let $P = \{a = x_0 < x_1 < \dots < x_m = b\}$ be a partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a bounded function

The upper-Darboux-sum of $f$ with respect to $P$ is

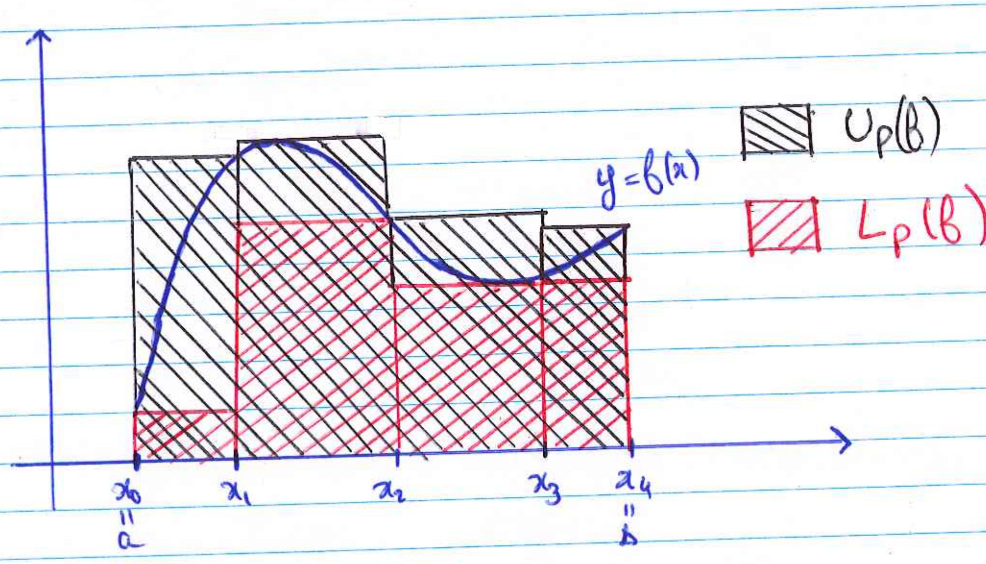
$$U_P(f) := \sum_{k=1}^{m} (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} (f)$$

$$= \sum_{I} \partial(I) \sup_{I} (f) \quad \text{where } I \text{ varies through the subintervals of } P$$

The lower-Darboux-sum of $f$ w.r.t. $P$ is

$$L_P(f) := \sum_{k=1}^{m} (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} (f)$$

$$= \sum_{I} \partial(I) \inf_{I} (f)$$




$$P = \{ \underset{\substack{\parallel \\ a}}{x_0} < x_1 < \dots < \underset{\substack{\parallel \\ b}}{x_4} \}$$

**Definition:** Let $P = (P_1, \dots, P_m)$ be a partition of $[a_1, b_1] \times \dots \times [a_m, b_m]$ and $f: [a_1, b_1] \times \dots \times [a_m, b_m] \longrightarrow \mathbb{R}$ be a bounded function

We define the upper-Darboux-sum of $f$ w.r.t $P$ by

$$U_P(f) = \sum_S \nu(S) \sup_S (f)$$

and the lower-Darboux-sum of $f$ w.r.t. $P$ by

$$L_P(f) = \sum_S \nu(S) \inf_S (f)$$

where $S$ varies through the subrectangles of $P$.

Rem: The assumption "$f$ bounded" ensures that $\sup_S f$ and $\inf_S f$ are well defined as the sup / inf of a non-empty bounded subset of $\mathbb{R}$

(Recall: $\sup_S f = \sup \{ f(x) : x \in S \}$, $\inf_S f = \inf \{ f(x) : x \in S \}$)

**Proposition:** $R$ = rectangle or segment, $f: R \to \mathbb{R}$ bounded, $P$ partition of $R$

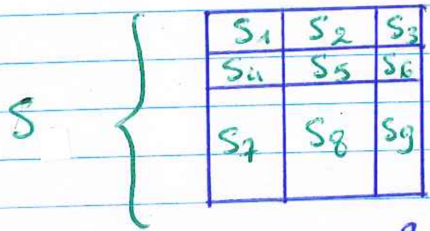
then $L_P(f) \leq U_P(f)$

$\Delta$ $L_P(f) = \sum_S \nu(S) \inf_S (f) \leq \sum_S \nu(S) \sup_S (f) = U_P(f)$ $\square$

**Proposition:** $R$ = rectangle or segment, $f: R \to \mathbb{R}$ bounded, $P, Q$ are 2 partitions of $R$

If $Q$ is finer than $P$ then $\begin{cases} U_Q(f) \leq U_P(f) \\ L_P(f) \leq L_Q(f) \end{cases}$

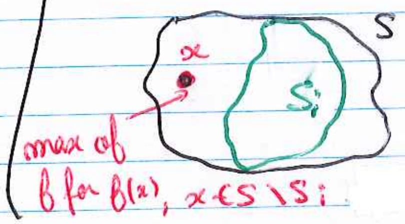
$\Delta$ Assume that $S$ is divided into subrectangles $S_1, \dots, S_9$




then $$\sum_{i=1}^{9} \partial(S_i) \sup_{S_i}(f) \leqslant \sum_{i=1}^{9} \partial(S_i) \sup_{S}(f)$$
$$= \left(\sum_{i=1}^{9} \partial(S_i)\right) \sup_S (f)$$
$$= \partial(S) \sup_S (f)$$

For this inequality, I used that $S_i \subset S \Rightarrow \sup_{S_i} \leqslant \sup_S$

i.e: when you shrink the domain, you may decrease the supremum




$\square$

<u>Corollary</u>: $f : R \to \mathbb{R}$ bounded, $R$ = rectangle or segment, $P, Q$ 2 partitions of $R$

$$L_P(f) \leqslant U_Q(f)$$

$\Delta$ Take $\Theta$ a partition of $R$ which is finer than $P$ and than $Q$

$$L_P(f) \leqslant L_\Theta(f) \leqslant U_\Theta(f) \leqslant U_Q(f)$$

$\square$

<u>Definition</u>: $R$ = a rectangle or segment line, $f : R \to \mathbb{R}$ bounded

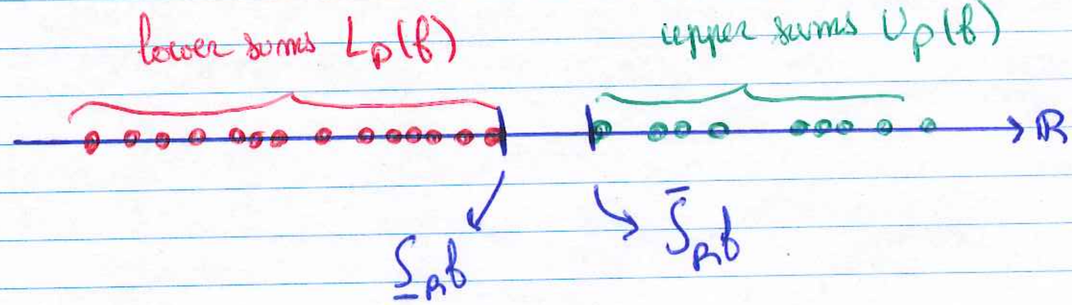
We define the Darboux lower integral of $f$ by

$$\underline{\int_R} f := \sup\{L_P(f) : P \text{ partition of } R\}$$

and the Darboux upper integral of $f$ by

$$\overline{\int_R} f := \inf\{U_P(f) : P \text{ partition of } R\}$$

<u>Remark</u>: they are always well-defined (as soon as $f$ is bounded)
indeed $\{U_P(f) : P \text{ partition of } R\}$ is not empty since it contains $U_{\{a<b\}}(f)$ for instance and it is bounded from below by $L_{\{a<b\}}(f)$ so $\inf\{U_P(f)\} =: \overline{\int}_R f$ exists and similarly for $\underline{\int}_R f$.




<u>Def</u>: $R :=$ rectangle or segment line, $f : R \longrightarrow \mathbb{R}$ bounded.
We say that $f$ is (Darboux) integrable if $\underline{\int}_R f = \overline{\int}_R f$ and then we define

$$\int_R f := \underline{\int}_R f = \overline{\int}_R f$$

<u>Theorem</u> ($\varepsilon$-criterion for integrability)
$R =$ rectangle or segment line, $f : R \longrightarrow \mathbb{R}$ bounded

$f$ is integrable $\Leftrightarrow$ $\forall \varepsilon > 0, \exists P$ partition of $R$ s.t. $U_P(f) - L_P(f) < \varepsilon$

$\Delta$ $\Rightarrow$: By assumption

$$\sup\{L_P(f)\} =: \underline{\int_R} f = \overline{\int_R} f := \inf\{U_P(f)\}$$

Let $\varepsilon > 0$.

Since $\overline{\int_R} f + \frac{\varepsilon}{2} > \overline{\int_R} f$ and $\overline{\int_R} f$ is the GLB of $\{U_P(f)\}$

$\overline{\int_R} f + \frac{\varepsilon}{2}$ is not a lower bound of $\{U_P(f)\}$,

ie $\exists P_1$ partition s.t. $U_{P_1}(f) < \overline{\int_R} f + \frac{\varepsilon}{2}$

Similarly $\exists P_2$ partition s.t. $L_{P_2}(f) > \overline{\int_R} f - \frac{\varepsilon}{2}$

Take $P$ a common refinement of $P_1$ and $P_2$ then

$$\begin{cases} U_P(f) \leq U_{P_1}(f) < \overline{\int_R} f + \frac{\varepsilon}{2} \\ L_P(f) \geq L_{P_2}(f) > \underline{\int_R} f - \frac{\varepsilon}{2} \end{cases}$$

$$\Rightarrow U_P(f) - L_P(f) < \cancel{\overline{\int_R} f} + \frac{\varepsilon}{2} - \cancel{\underline{\int_R} f} + \frac{\varepsilon}{2} = \varepsilon$$

are equal since $f$ is integrable

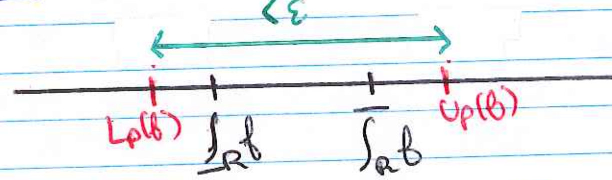
$\Leftarrow$: Let $\varepsilon > 0$, then $\exists$ a partition s.t. $U_P(f) - L_P(f) < \varepsilon$

But $L_P(f) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq U_P(f)$

Hence $0 \leq \overline{\int_R} f - \underline{\int_R} f \leq U_P(f) - L_P(f) < \varepsilon$

ie: $\forall \varepsilon > 0$, $0 \leq \overline{\int_R} f - \underline{\int_R} f < \varepsilon$

$\Rightarrow \overline{\int_R} f = \underline{\int_R} f$




$\square$

**Theorem:** $f, g : R \to \mathbb{R}$ integrable, $c \in \mathbb{R}$ then

① $(f+g) : R \to \mathbb{R}$ is integrable and

$$\int_R (f+g) = \int_R f + \int_R g$$

② $(cf) : R \to \mathbb{R}$ is integrable and

$$\int_R (cf) = c \int_R f$$

③ $(fg) : R \to \mathbb{R}$ is integrable and

$$\left[\int_R (fg)\right]^2 \leq \int_R f^2 \int_R g^2$$

"Cauchy-Schwarz inequality"

④ If $\forall x \in R$, $f(x) \leq g(x)$

then $$\int_R f \leq \int_R g$$

⑤ $|f| : R \to \mathbb{R}$ is integrable and

$$\left| \int_R f \right| \leq \int_R |f|$$

⚠

- $\int (fg) \neq \int f \cdot \int g$ ; eg: $f(x) = \begin{cases} 1 & \text{on } [0,1/2) \\ 0 & \text{on } [1/2,1] \end{cases}$ $g(x) = \begin{cases} 1 & \text{on } (1/2,1] \\ 0 & \text{on } [0,1/2] \end{cases}$
- $|f|$ integrable $\not\Rightarrow$ $f$ integrable ; eg $f(x) = \begin{cases} 1 & \text{on } \mathbb{Q} \cap [0,1] \\ -1 & \text{on } (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \end{cases}$
- $\int |f| \neq \left| \int f \right|$ : eg $f(x) = x$ on $[-1,1]$

The proofs may be difficult / technical if you directly start computing the upper/lower integrals. Below are some strategies to simplify the proofs.

## Δ Sketch of proof:

① $\int f + \int g = \underline{\int} f + \underline{\int} g$ ← since $f, g$ are integrable

$\leq \underline{\int} (f+g)$ ← Prove it from the definition of $L_P(f)$

$\leq \overline{\int} (f+g)$

$\leq \overline{\int} f + \overline{\int} g$

$= \int f + \int g$ ← since $f, g$ are integrable

$\Rightarrow \underline{\int} (f+g) = \overline{\int} (f+g)$

② If $c \geq 0$: $\underline{\int} (cf) = c \underline{\int} f = c \overline{\int} f = \overline{\int} (cf)$

since $f$ is integrable

prove it using the Darboux sums

To get the result for $c<0$:

$\underline{\int} (-f) = -\overline{\int} f = -\underline{\int} f = \overline{\int} (-f)$

since $f$ is integrable

since $\sup(-f) = -\inf(f)$

since $\inf(-f) = -\sup(f)$

③ We first prove that $f^2$ is integrable:

$|f(x)^2 - f(y)^2| = |f(x) + f(y)|\,|f(x) - f(y)|$

$\leq (|f(x)| + |f(y)|)\,|f(x) - f(y)|$

$\leq 2M\,|f(x) - f(y)|$ since $f$ is bounded: $\exists M > 0, \forall x, |f(x)| \leq M$

$\Rightarrow \sup_S (f^2) - \inf_S (f^2) \leq 2M (\sup_S f - \inf_S f)$ where $S$ is a rectangle

$\Rightarrow U_P(f^2) - U_L(f^2) \leq 2M (U_P(f) - L_P(f))$

and we conclude with the $\varepsilon$-criterion.

then notice that

$$(fg) = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$ which is integrable by ①, ② and the above

We may adapt the proof of the usual CS inequality (Sep 5)

④ $\int g - \int f = \int (g-f) = \overline{\int}(\underbrace{g-f}_{\geq 0}) \geq 0$

prove it using the definition. $\forall p,\ U_p \geq 0 \Rightarrow \overline{\int} \geq 0$

⑤ From the reverse triangle inequality:

$$\forall x, y, \quad |f(x)| - |f(y)| \leq |f(x) - f(y)|$$

$$\Rightarrow \quad \sup_S |f| - \inf_S |f| \leq \sup f - \inf f$$

$$\Rightarrow \quad U_p(|f|) - L_p(|f|) \leq U_p(f) - L_p(f)$$

and we conclude with the $\varepsilon$-criterion

For the remaining inequality:

$$|f| \geq f \Rightarrow \int |f| \geq \int f$$

$$|f| \geq -f \Rightarrow \int |f| \geq \int (-f) = -\int f$$

Hence $\int |f| \geq \left| \int f \right|$

$\square$

<u>Theorem</u> : $R$ = rectangle or segment line

If $f : R \to \mathbb{R}$ is continuous then $f$ is integrable

$\Delta$. First notice that $f$ is bounded as a continuous function defined on a compact set : $\begin{cases} R \text{ compact} \\ f \ C^0 \end{cases} \Rightarrow f(R)$ compact $\Rightarrow f(R)$ bounded

- Next, since $f$ is continuous on $R$ compact, by Heine-Cantor theorem $f$ is uniformly continuous :

Let $\varepsilon > 0$,

$$\exists \delta > 0, \ \forall x_1, x_2 \in R, \quad \|x_1 - x_2\| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\varepsilon}{2 v(R)}$$

Let $P$ be a partition of $R$ such that for any subrectangle $S$,

$$x_1, x_2 \in S \Rightarrow \|x_1 - x_2\| < \delta$$

Then

$$U_P(f) - L_P(f) = \sum_S v(S) \left( \sup_S f - \inf_S f \right)$$

$$\leqslant \sum_S v(S) \cdot \frac{\varepsilon}{2 v(R)}$$

$$= \frac{\varepsilon}{2 v(R)} \cdot \sum_S v(S)$$

$$= \frac{\varepsilon}{2} \frac{v(R)}{v(R)}$$

$$= \frac{\varepsilon}{2} < \varepsilon$$

So $f$ is integrable by the $\varepsilon$-criterion $\square$

# The FTC - Recollection from MAT137

Let $a < c < b$ and $f : [a,b] \to \mathbb{R}$ integrable then $f : [a,c] \to \mathbb{R}$ and $f : [c,b] \to \mathbb{R}$ are too and $\int_a^b f = \int_a^c f + \int_c^b f$

Hence it is natural to define $\int_b^a f := -\int_a^b f$ when $a > b$

__Theorem:__ If $f : [a,b] \to \mathbb{R}$ continuous then $\exists c \in [a,b]$ s.t. $\int_a^b f = (b-a) f(c)$

$\Delta$ $f$ is continuous on a compact hence $\exists s, S \in [a,b]$ s.t. $\forall x \in [a,b], f(s) \leq f(x) \leq f(S)$

$$\Rightarrow \underbrace{\int_a^b f(s)\,dx}_{= f(s)(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b f(S)\,dx}_{= f(S)(b-a)}$$

$$\Rightarrow f(s) \leq \frac{\int_a^b f}{b-a} \leq f(S)$$

We conclude with the IVT $\square$

__Theorem__ (FTC - Part 1)

Let $I$ interval, $f : I \to \mathbb{R}$ continuous, and $a \in I$.

Define $F : I \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\,dt$

Then $F$ is differentiable and $F' = f$

$\Delta$ • $f$ is continuous on $[a,x]$ if $x \geq a$ or $[x,a]$ otherwise so $F$ is well-defined

• Let $x_0 \in I$ then

$$F(x) - F(x_0) = \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt = \int_x^{x_0} f(t)\,dt = (x - x_0) f(\xi)$$

for some $\xi \in [x, x_0]$ or $[x_0, x]$ by the above theorem

$$\Rightarrow \frac{F(x) - F(x_0)}{x - x_0} = f(\xi) \xrightarrow[\substack{x \to x_0 \\ \Rightarrow \xi \to x_0}]{} f(x_0) \text{ since } f \text{ is } C^0$$

$$\Rightarrow F'(x_0) = f(x_0)$$

$\square$

**Cor**: Let $f: I \to \mathbb{R}$ be a $C^0$ function defined on an interval, $a \in I$.
If $F: I \to \mathbb{R}$ is an antiderivative of $f$
then $\exists\, C \in \mathbb{R}$ s.t. $F(x) = \int_a^x f(t)\,dt + C$

$\Delta$ $F$ and $x \mapsto \int_a^x f$ are two antiderivatives of $f$ on an interval
hence they differ by a constant by the MVT $\square$

Remark: If the domain is not an interval, it's possible to find two antiderivatives which don't differ by a const

eg: $F_1, F_2 : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $F_1(x) = \ln|x|$, $F_2(x) = \begin{cases} \ln|x| + 42 & \text{for } x > 0 \\ \ln|x| - \pi & \text{for } x < 0 \end{cases}$

then $F_1' = F_2' = f$ for $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ $f(x) = \frac{1}{x}$

but $F_1 - F_2 \neq$ constant

Theorem: (FTC - part 2) $f: [a,b] \to \mathbb{R}$ $C^0$, $F: [a,b] \to \mathbb{R}$ an antiderivative of $f$

then $\int_a^b f(t)\,dt = F(b) - F(a)$

$\Delta$ By the above $F(x) = \int_a^x f(t)\,dt + C$ for some $C$
hence $F(b) - F(a) = \int_a^b f(t)\,dt + C - \int_a^a f(t)\,dt - C = \int_a^b f(t)\,dt$ $\square$

Remark: we may replace $f$ $C^0$ by $f$ integrable in the above but the proof becomes a little bit more technical

⚠ If $f$ is integrable but not $C^0$, $F(x) = \int_a^x f(t)\,dt$ may not be differentiable (but it is uniformly $C^0$)

⚠ It is not enough to have an antiderivative to be integrable
Eg: $F(x) = \begin{cases} x^2 \sin(\pi/x^2) & \text{on } (0,1] \\ 0 & \text{at } 0 \end{cases}$ : $F$ is differentiable but $F'$ is not integrable ($F'$ is not bounded)

## Zero content sets

**Def.** We say that $S \subset \mathbb{R}^n$ has zero content if for every $\varepsilon > 0$ there exist finitely many rectangles (or segment lines if $n=1$) $R_1, \dots, R_q$ such that

① $\sum_{i=1}^{q} \vartheta(R_i) < \varepsilon$

② $S \subset \bigcup_{i=1}^{q} R_i$

Exercises • $S$ has content zero $\Rightarrow S$ is bounded
• $[0,1]$ doesn't have zero content

**Proposition.** ① $\left.\begin{array}{l}\tilde{S} \subset S \\ S \text{ has zero content}\end{array}\right\} \Rightarrow \tilde{S}$ has zero content

② $S$ has zero content $\Leftrightarrow \overline{S}$ has zero content

③ $S_1, \dots, S_r$ have zero content $\Rightarrow S = \bigcup_{i=1}^{r} S_i$ has zero content

④ $S$ finite $\Rightarrow S$ has zero content

⑤ Let $R \subset \mathbb{R}^n$ be a rectangle (or segment line for $n=1$). If $f : R \to \mathbb{R}$ is integrable then the graph of $f$
$$\Gamma_f = \{(x, f(x)) : x \in R\} \subset \mathbb{R}^{n+1}$$
has zero content

$\Delta$ ① Let $\varepsilon > 0$, since $S$ has zero content, there exist finitely many rectangles $R_1, \dots, R_q$ s.t.

① $\sum_{i=1}^{q} \vartheta(R_i) < \varepsilon$ ② $\tilde{S} \subset S \subset \bigcup_{i=1}^{q} R_i$

hence $\tilde{S}$ has zero content

② $\Leftarrow$: if $\overline{S}$ has zero content then $S \subset \overline{S}$ has too by ①
$\Rightarrow$: assume that $S$ has zero content and let $\varepsilon > 0$. Then $\exists$ rectangles $R_1, \dots, R_q$ s.t. $\sum \vartheta(R_i) < \varepsilon$ and $S \subset \bigcup R_i \Rightarrow \overline{S} \subset \overline{\bigcup R_i} = \bigcup R_i$ since $\bigcup R_i$ is closed as a finite union of closed sets. Hence $\overline{S}$ has zero content

③ Let $\varepsilon > 0$. Since $S_i$ has zero content, $\exists R_1^i, \dots, R_{q_i}^i$ rectangles s.t.

$$\sum_{j=1}^{q_i} \mathcal{V}(R_j^i) < \frac{\varepsilon}{r} \quad \text{and} \quad S_i \subset \bigcup_{j=1}^{q_i} R_j^i$$

then $S = \bigcup_{i=1}^{r} S_i \subset \bigcup_{i=1}^{r} \bigcup_{j=1}^{q_i} R_j^i$ and $\sum_{i=1}^{r} \sum_{j=1}^{q_i} \mathcal{V}(R_j^i) < \sum_{i=1}^{r} \frac{\varepsilon}{r} = \varepsilon$

hence $S$ has zero content

④ First assume that $S = \{p\}$ has only one element $p = (x_1, \dots, x_m)$

Let $\varepsilon > 0$, set $\delta = \sqrt[m]{\varepsilon/2}$

Set $R = [x_1 - \delta/2, x_1 + \delta/2] \times \dots \times [x_m - \delta/2, x_m + \delta/2]$

then $S \subset R$ and $\mathcal{V}(R) = \delta^m = \varepsilon/2 < \varepsilon$

hence $S$ has zero content

Finally $S = \{p_1, \dots, p_r\} = \bigcup_{i=1}^{r} \{p_i\}$ has zero content by ③

⑤ Let $\varepsilon > 0$. Since $f$ is integrable there exists a partition $P$ s.t.

$$\sum_{S} (\sup_S f - \inf_S f)\, \mathcal{V}(S) < \varepsilon$$ where $S$ goes through the subrectangles of $P$

finitely many

Notice that $\Gamma_f \subset \bigcup_S \big( \underbrace{[\inf_S f, \sup_S f] \times S}_{\text{rectangle}} \big)$

and $\sum_S \mathcal{V}([\inf_S f, \sup_S f] \times S) = \sum_S (\sup_S f - \inf_S f)\mathcal{V}(S) < \varepsilon$ $\square$

<u>Remark</u>: the converse of ⑤ is $\boxed{\text{false}}$. ⚠

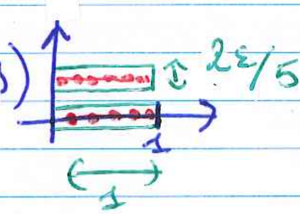
Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

then $f$ is not integrable.

Let $\varepsilon > 0$,

then $\Gamma_f \subset \overbrace{([0,1] \times [1-\varepsilon/5, 1+\varepsilon/5])}^{R_1} \cup \overbrace{([0,1] \times [-\varepsilon/5, \varepsilon/5])}^{R_2}$

and $\mathcal{V}(R_1) + \mathcal{V}(R_2) = \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} < \varepsilon$




hence $\Gamma_f$ has zero content.

**Theorem** : $R$ = rectangle or segment line, $f : R \to \mathbb{R}$ **bounded**

If $\{x \in R : f \text{ is not continuous at } x\}$ has zero content

then $f$ is integrable

$\Delta$ Let $\varepsilon > 0$.

By assumption $\{x \in R : f \text{ is not continuous at } x\}$ has zero content

hence it may be covered by rectangles $S_1, \dots, S_q$ s.t.

(**) $$\sum_i \mathcal{V}(S_i) < \frac{\varepsilon}{2(\sup_R f - \inf_R f)}$$ Here I assume that $f$ is not constant otherwise it is obvious.

Without loss of generality, we may assume that they are subrectangles of a partition $P$ of $R$ and that the other subrectangles don't contain discontinuity points

Now, since the union $K$ of the remaining subrectangles is compact,

$f$ is uniformly continuous on $K$.

Hence we may refine the partition s.t. $\sum\limits_{S \text{ has no disc point}} (\sup_S f - \inf_S f) \cdot \mathcal{V}(S) < \frac{\varepsilon}{2}$ (*)

(as in the proof "$C^0 \Rightarrow$ integrable")

then $\sum\limits_S (\sup_S f - \inf_S f) \cdot \mathcal{V}(S)$

$$= \sum_{S \text{ has no disc}} (\sup_S f - \inf_S f)\mathcal{V}(S) + \sum_{S \text{ has disc}} (\sup_S f - \inf_S f) \cdot \mathcal{V}(S)$$

$$= \qquad \text{"} \qquad + \sum (\sup_R f - \inf_R f) \cdot \mathcal{V}(S)$$

$$< \quad \underbrace{\varepsilon/2}_{\text{by } (*)} \quad + \underbrace{\varepsilon/2}_{\text{by } (**)}$$

Hence $f$ is integrable $\square$

<u>Remark</u>: the converse is false!

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q,\ p \in \mathbb{Z}\setminus\{0\},\ q \in \mathbb{N}_{>0},\ \gcd(p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Then • $f$ is integrable

• The discontinuity set of $f$ is $[0,1] \cap \mathbb{Q}$

But • $[0,1] \cap \mathbb{Q}$ doesn't have zero content

indeed, assume by contradiction that $[0,1] \cap \mathbb{Q}$ has zero content

then $\overline{[0,1] \cap \mathbb{Q}} = [0,1]$ has zero content: contradiction. $\Delta$

x—x

<u>Remark</u> (NOT PART OF MAT 237)

If we replace "finitely many" by "countably many" in the definition of zero content set then we obtain the definition of "set of measure 0"

Then we have Lebesgue Criterion for Riemann's integrability:

<u>Theorem</u>: $f : R \to \mathbb{R}$ <u>bounded</u> on a rectangle/segment line

$f$ is integrable $\iff \{x \in R : f \text{ is discontinuous}\}$ has measure 0

↳ <u>both way now!</u>

How to integrate on a set which is not a rectangle?

Let $S \subset \mathbb{R}^n$ be a bounded subset

We define the characteristic function of $S$ by

$\chi_S : \mathbb{R}^n \to \mathbb{R}$, $\chi_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ otherwise} \end{cases}$

Let $f: S \to \mathbb{R}$ be a function

We say that $f$ is integrable on $S$ if there exists a rectangle $R \subset \mathbb{R}^n$ such that $S \subset R$ and $f\chi_S : R \to \mathbb{R}$ is integrable

We write $\int_S f = \int_R f\chi_S$

Remarks: ① $f\chi_S$ is an abuse of notation for the function

$$\begin{aligned} R &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} f(x) \text{ if } x \in S \\ 0 \quad \text{if } x \in R \setminus S \end{cases} \end{aligned}$$

② One may check that this definition is independent of the choice of the rectangle $R$: if $R'$ is another suitable rectangle then $\int_R f\chi_S = \int_{R'} f\chi_S$

③ The basic properties ($f+g$, $cf$, $|f|$) remain true

④ $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$

⑤ Even if $f$ is $C^0$, there is no reason for $\chi_S f$ to be (look at points on $\partial S$)

__Exercise 1__ : $\chi_S$ is discontinuous at $x \Leftrightarrow x \in \partial S$

__Exercise 2__ : $S \subset \mathbb{R}^m$ bounded, $f : S \to \mathbb{R}$ bounded

If $\begin{cases} \partial S \text{ has zero content} \\ \{x \in S : f \text{ is not } C^0 \text{ at } x\} \text{ has zero content} \end{cases}$

then $f$ is integrable on $S$, __ie__ $\int_S f$ is well-defined

__Exercise 3__ : If $S$ has zero content and $f : S \to \mathbb{R}$ is bounded

then $f$ is integrable on $S$ and $\int_S f = 0$

__Exercise 4__ : $f, g : S \to \mathbb{R}$ are integrable

If $\{x \in S : f(x) \neq g(x)\}$ has zero content then $\int_S f = \int_S g$

__Exercise 5__ : If $f$ is integrable on $S$ and on $T$ and $S \cap T$ has zero content then $f$ is integrable on $S \cup T$ and

$$\int_{S \cup T} f = \int_S f + \int_T f$$

$\partial S$ bounded $\not\Rightarrow$ $S$ bounded

__Ex__ : $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ is not bounded

but $\partial S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is bounded

# Solutions

**Ex1**: • If $x_0 \in \mathring{S}$, $\exists \varepsilon > 0$, $B(x_0, \varepsilon) \subset \mathring{S} \subset S$

and $f$ is constant equal $1$ on this ball

• If $x_0 \in (\overline{S})^c = (S^c)^\circ$ then $\exists \varepsilon > 0$, $B(x_0, \varepsilon) \subset (S^c)^\circ \subset S^c$

and $f$ is constant equal $0$ on this ball

• If $x_0 \in \partial S$ then for any $\delta > 0$,

$$\left.\begin{array}{l} \exists y_1 \in B(x_0, \delta) \text{ s.t. } f(y_1) = 0 \\ \exists y_2 \in B(x_0, \delta) \text{ s.t. } f(y_2) = 1 \end{array}\right\} \text{prove it! using the definition of } \partial S$$

**Ex2**: Hint: the set of discontinuity of $\chi_S f : R \to \mathbb{R}$

is a subset of $\underbrace{\partial S}_{\text{zero content}} \cup \underbrace{\{x \in S : f \text{ not } C^0 \text{ at } x\}}_{\text{zero content}}$

where $R$ is a rectangle containing $S$

**Ex3**: Let $\varepsilon > 0$. $M = \sup |f|$.

$\exists R_1, \dots, R_q$ rectangle s.t. $S \subset \bigcup_{i=1}^{q} R_i$ and $\sum_{i=1}^{q} \mathcal{V}(R_i) < \varepsilon/2M$

We may assume $R_1, \dots, R_q$ are subrectangles of a partition $P$ of a rectangle $R$ containing $S$.

Then, for $\chi_S f : R \to \mathbb{R}$

$$-\frac{\varepsilon}{2} = -\frac{\varepsilon M}{2M} < \sum_{i=1}^{q} \mathcal{V}(R_i)(-M) \leq L_P(\chi_S f) \leq U_P(\chi_S f) \leq \sum_{i=1}^{q} \mathcal{V}(R_i)\, M \leq \frac{\varepsilon M}{2M} = \varepsilon/2$$

$$\Rightarrow U_P(\chi_S f) - L_P(\chi_S f) < \varepsilon$$

So $\chi_S f$ is integrable on $R$ and $f$ is integrable on $S$

Ex 4: $h = f - g$ is zero except on a set $T$ which has zero content

So $h$ is integrable and $\int_S h = 0$ by the previous exo

$$\int_S h = \int_S (f-g) = \int_S f - \int_S g$$

since they are both integrable

$$\Rightarrow \int_S f = \int_S g$$

Ex 5: $\chi_{S \cup T} f = \chi_S f + \chi_T f - \chi_{S \cap T} f$

integrable by assumption

integrable by Exo 3 since $S \cap T$ has zero content

So $\chi_{S \cup T} f$ is integrable

and $\int_{S \cup T} f = \int \chi_{S \cup T} f = \int \chi_S f + \int \chi_T f - \int \chi_{S \cap T} f$

$= \int_S f + \int_T f - 0$