Uniform continuity:

In what follows: $S \subset \mathbb{R}^n$, $f : S \longrightarrow \mathbb{R}^p$

Definition: $x_0 \in S$.
We say that f is continuous at x_0 if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$$

Definition: We say that f is continuous if it is everywhere, ie:

$$\forall x_0 \in S, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$$

Definition: We say that f is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$$

Let's compare these two definitions.

Continuity

① $\forall \varepsilon > 0, \boxed{\forall x_1 \in S, \exists \delta > 0}, \forall x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$

② $\forall \varepsilon > 0, \boxed{\exists \delta > 0, \forall x_1 \in S}, \forall x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$

Uniform continuity

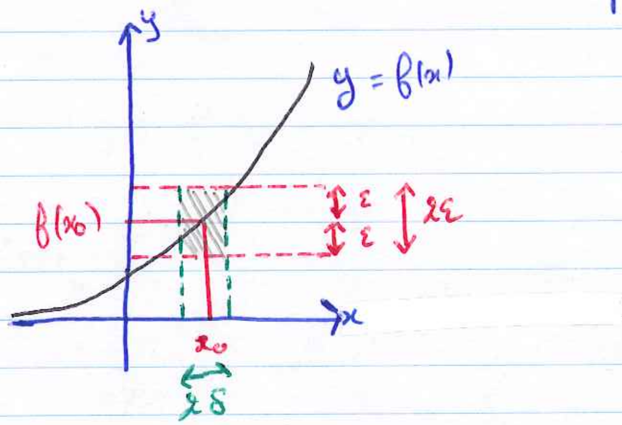
The only difference is that in ① δ may depend on the choice of x_1
but in ② δ is independant of x_1 or x_2:
it should be suitable everywhere

Continuity is a local notion (only depends on the behavior of f around x_0)

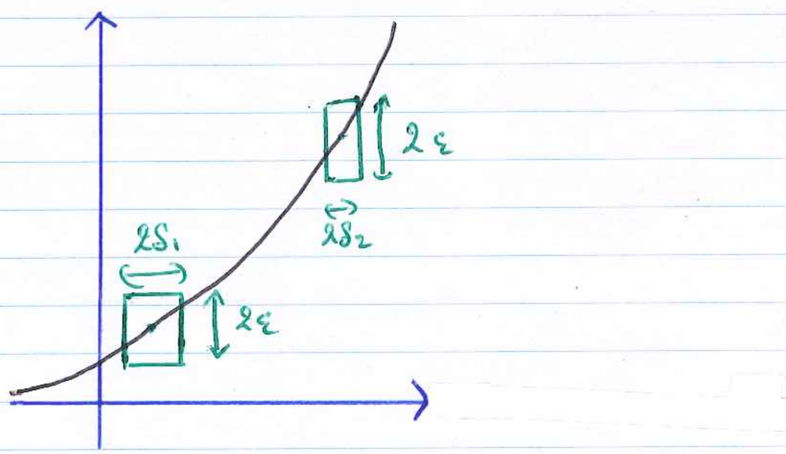
Uniform continuity is a global notion (depends on the domain)

Remark: obviously: uniform continuity implies continuity.

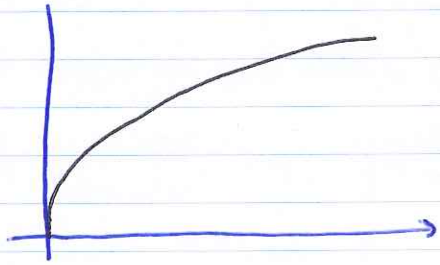
Geometrically: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is continuous since for any $x_0 \in \mathbb{R}$, and any $\varepsilon > 0$, you may find a $\delta > 0$ s.t. the graph of f stays in the $2\delta \times 2\varepsilon$ box around $(x_0, f(x_0))$ for $x \in (x_0 - \delta, x_0 + \delta)$



But f is not uniformly continuous: the more x_0 goes , the smaller must be δ (for a fixed $\varepsilon > 0$)
$\rightsquigarrow$ we can't find a δ suitable for everywhere



However $g : [0, +\infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$ is uniformly continuous even if $\lim_{x \to +\infty} g(x) = +\infty$ and the graph becomes arbitrarily steeps at 0.



If you recall the discussion about Dedekind-Completeness from Sep 24, the below stated theorem is another characterization of the Dedekind completeness of $\mathbb{R}$

You can safely skip this comment

The Heine-Cantor theorem :

$K \subset \mathbb{R}^n$ compact and $f : K \longrightarrow \mathbb{R}^p$

If f is continuous then f is uniformly continuous

Δ We prove the contrapositive :

f not uniformly continuous $\Rightarrow$ f not continuous.

Let's assume that f is not U.C.

$\exists \varepsilon > 0, \forall n \in \mathbb{N}_{>0}, \exists x_n^1, x_n^2 \in K, \|x_n^1 - x_n^2\| < \frac{1}{n}$ and $\|f(x_n^1) - f(x_n^2)\| \geq \varepsilon$

(x_n^1) is a sequence of terms in K compact so $\exists$ a subsequence $(x^1_{\varphi(n)})$ convergent to $l \in K$

$$\begin{aligned} \|x^2_{\varphi(n)} - l\| &= \|x^2_{\varphi(n)} - x^1_{\varphi(n)} + x^1_{\varphi(n)} - l\| \\ &\leq \|x^2_{\varphi(n)} - x^1_{\varphi(n)}\| + \|x^1_{\varphi(n)} - l\| \\ &\leq \frac{1}{\varphi(n)} + \|x^1_{\varphi(n)} - l\| \xrightarrow[n \to \infty]{} 0 + 0 = 0 \end{aligned}$$

so $\lim\limits_{n \to +\infty} x^2_{\varphi(n)} = l$ too

Assume by contradiction that f is continuous at $l \in K$

then $\forall n$, $\|f(x^1_{\varphi(n)}) - f(x^2_{\varphi(n)})\| \geq \varepsilon$

$\Rightarrow$ $\|f(l) - f(l)\| \geq \varepsilon$ by continuity of f and $\|\cdot\|$

ie $0 \geq \varepsilon > 0$

Contradiction.

Hence f is not continuous at l $\square$

A few exercises to practice U.C.

Ex 1: I interval, $f : I \to \mathbb{R}$

Prove that: f Lipschitz $\Rightarrow$ f U.C.

Ex 2: $I = (a,b)$, $a \in \mathbb{R}$, $b = \mathbb{R} \cup \{+\infty\}$
$f : I \to \mathbb{R}$

① Prove that: f U.C $\Rightarrow \lim\limits_{x \to a^+} f(x)$ exists and is finite

② Deduce that: $\lim\limits_{x \to a^+} f(x)$ DNE $\Rightarrow$ f is not U.C.

Ex 3: $f : [0,+\infty) \to \mathbb{R}$

① Prove that f UC $\Rightarrow \exists a,b \in \mathbb{R}, \forall x \in [0,+\infty), f(x) \leqslant ax+b$

Remark: we have a similar result on $(-\infty, 0]$, but not if the domain is $\mathbb{R}$ entirely (eg: $f(x) = |x|$ is U.C. but not "upper bounded" by an affine function)

② Deduce that if $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = +\infty$ then f is not U.C.

③ Then if $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = -\infty$ then f is not U.C.

Ex 4: $f : [a,+\infty) \to \mathbb{R}$: If $\begin{cases} f \text{ is continuous} \\ \text{and} \\ \lim\limits_{x \to +\infty} f(x) \in \mathbb{R} \\ \quad\quad \text{"}\ell \end{cases}$ then f is U.C.

Ex 5: Prove that

① $x^2 : \mathbb{R} \to \mathbb{R}$ is not U.C.

② $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is not U.C.

③ $\frac{1}{x} : (0,+\infty) \to \mathbb{R}$ is not U.C

④ $\exp : \mathbb{R} \to \mathbb{R}$ is not U.C.

⑤ $\exp : [-\pi, 42] \to \mathbb{R}$ is U.C.

⑥ $\sqrt{\cdot} : [0,+\infty) \to \mathbb{R}$ is U.C.

⑦ $\sqrt[3]{\cdot} : \mathbb{R} \to \mathbb{R}$ is U.C.

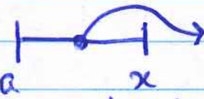
⑧ $\sin : \mathbb{R} \to \mathbb{R}$ is U.C.

⑨ $\sin(x^2) : \mathbb{R} \to \mathbb{R}$ is not U.C

⑩ $\sin(\frac{1}{x}) : (0,1) \to \mathbb{R}$ is not U.C.

Ex2 ① Check that if (x_n) is a sequence of I converging to a then $(f(x_n))$ is Cauchy
Hence $\ell = \lim f(x_n)$ exists

Then prove that $\lim\limits_{x \to a^+} f(x) = \ell$

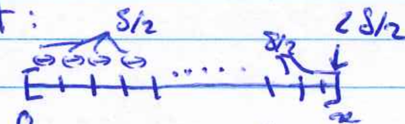
Hint:  x_n for n big enough: $|a-x_n| < |a-x| < \delta$

So $|f(x)-\ell| \leq |f(x)-f(x_n)| + |f(x_n)-\ell| < \underbrace{\frac{\varepsilon}{2}}_{\text{UC}} + \underbrace{\varepsilon/2}_{\lim}$

② Contrapositive

Ex3 ① We know that $\exists \delta > 0, \forall x, y \in [0,+\infty), |x-y| < \delta \Rightarrow |f(x)-f(y)| < 1$ (∗)

Let $x \in [0,+\infty)$. Divide $[0,x]$ in a partition such that:
$\forall k \in [0, n-2],\ x_{k+1} - x_k = \delta/2,\ x_{n} - x_{n-1} < \delta/2$:



$$|f(x)-f(0)| = \left|\sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k)\right| \leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$$

$$(n-1)\delta/2 < x \xrightarrow{(*)} \leq n < \frac{2}{\delta}x + 1$$

Hence $f(x) \leq \frac{2}{\delta}x + 1 + f(0)$

② By contrapositive: f UC $\Rightarrow f(x) \leq ax+b \Rightarrow \frac{f(x)}{x} \leq a + b/x$

③ Replace f by $-f$

Ex4: Let $\varepsilon > 0$, $\exists A > 0$ st. $\forall x \in [a,+\infty)$, $x \geq A \Rightarrow |f(x)-\ell| < \varepsilon/2$ (∗)

Heine Cantor on $[a,A]$: $\exists \delta > 0, \forall x,y \in [a,A], |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2$ (∗∗)

Let $x, y \in [a,+\infty)$

Case1. $x,y \geq A$ then $|f(x)-f(y)| = |f(x)-\ell+\ell-f(y)| \leq |f(x)-\ell| + |f(y)-\ell| \underset{\text{by } (*)}{<} \frac{\varepsilon}{2} + \varepsilon/2$

Case2. $x,y \in [a,A]$, $|f(x)-f(y)| < \varepsilon/2 < \varepsilon$ by (∗∗)

Case3. $x \in [a,A]$, $y \in [A,+\infty)$, $|f(x)-f(y)| = |f(x)-f(A)+f(A)-f(y)|$
$\leq |f(x)-f(A)| + |f(A)-f(y)|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (∗∗) and (∗)

Ex 5 → some hints only

⑥ • $\sqrt{\cdot}$ is U.C. on $[0,1]$ by Heine-Cantor

• if $x, y \geq 1$ then $|\sqrt{x} - \sqrt{y}| = \dfrac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \dfrac{|x-y|}{2}$

so $\sqrt{\cdot}$ is U.C. on $[1, +\infty)$

then "patch" as in Ex 4 (Case 1: $x, y \in [0,1]$ Case 2: $x, y \geq 1$ Case 3: $x \in [0,1], y \geq 1$)

⑨ $x_n = \sqrt{n\pi + \frac{\pi}{2}}$ $y_n = \sqrt{n\pi}$

then $x_n - y_n = \sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} = \dfrac{n\pi + \frac{\pi}{2} - \pi n}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} = \dfrac{\pi/2}{\sqrt{n\pi + \pi/2} + \sqrt{n\pi}} \xrightarrow[n\to\infty]{} 0$

but $\sin(x_n^2) = \pm 1$ & $\sin(y_n^2) = 0$

⑦ Similar to ⑥

② Method 1: $x_n = \frac{\pi}{2} - \frac{1}{n}$, $y_n = \frac{\pi}{2} - \frac{1}{2n}$, $|y_n - x_n| \xrightarrow[n\infty]{} 0$

but $|f(x_n) - f(y_n)| = \dfrac{1}{\sin(1/n)} \geq 1$

Method 2: Use Exo 2

For the others: use the previous exercises + Heine-Cantor