The Implicit Function Theorem

Recollection:

$\triangle$ $x = (x_1, \dots, x_m)$
$y = (y_1, \dots, y_p)$
$F = (F_1, \dots, F_p)$

Theorem: $U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^p$ open

$$F : U \times V \longrightarrow \mathbb{R}^p \quad \text{of class } C^1$$
$$(x, y) \longmapsto F(x, y)$$

$$(x_0, y_0) \in U \times V$$

If $D_y F(x_0, y_0)$ is invertible then there exist $r, s > 0$ satisfying $B(x_0, r) \subset U$, $B(y_0, s) \subset V$ and $\varphi : B(x_0, r) \longrightarrow B(y_0, s)$ of class C^1 such that

(*) $$\forall (x, y) \in B(x_0, r) \times B(y_0, s), \quad F(x, y) = F(x_0, y_0) \Leftrightarrow y = \varphi(x)$$

Remember that $D_y F(x_0, y_0)$ is the Jacobian matrix of $y \mapsto F(x_0, y)$ at y_0 and that $D_x F(x_0, y_0)$ is the Jacobian matrix of $x \mapsto F(x, y_0)$ at x_0

$$\underset{\substack{\cap \\ M_{p, m+p}(\mathbb{R})}}{DF(x_0, y_0)} = \left(\underbrace{\begin{matrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial x_m}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_p}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_p}{\partial x_m}(x_0, y_0) \end{matrix}}_{\substack{= \\ D_x F(x_0, y_0) \\ \cap \\ M_{p,m}(\mathbb{R})}} \; \underbrace{\begin{matrix} \frac{\partial F_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y_p}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_p}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_p}{\partial y_p}(x_0, y_0) \end{matrix}}_{\substack{= \\ D_y F(x_0, y_0) \\ \cap \\ M_{p,p}(\mathbb{R})}} \right)$$

__Remark:__ $F(x_0,y_0) = F(x_0,y_0)$ so $y_0 = \varphi(x_0)$ by $(*)$

__Remark:__ $F(x,\varphi(x)) = F(x_0,y_0) \quad \forall x \in B(x_0,r)$

$$\Rightarrow DF(x_0, \underbrace{\varphi(x_0)}_{= y_0}) \begin{pmatrix} I_{m,m} \\ D\varphi(x_0) \end{pmatrix} = 0$$

↳ the RHS is constant

↳ by the chain rule applied to $F \circ G(x)$ where $G(x) = (x, \varphi(x))$

$$\Rightarrow \begin{pmatrix} D_xF(x_0,y_0) & D_yF(x_0,y_0) \end{pmatrix} \begin{pmatrix} I_{m,m} \\ D\varphi(x_0) \end{pmatrix} = 0$$

$$\Rightarrow D_xF(x_0,y_0) + D_yF(x_0,y_0)\, D\varphi(x_0) = 0$$

$$\Rightarrow D\varphi(x_0) = -\left[D_yF(x_0,y_0)\right]^{-1} D_xF(x_0,y_0)$$

recall that $D_yF(x_0,y_0)$ is invertible

__Ccl:__ We know how to compute $D\varphi(x_0)$ in terms of F

$$\boxed{D\varphi(x_0) = -\left[D_yF(x_0,y_0)\right]^{-1} D_xF(x_0,y_0)}$$

You should know this formula (or better: be able to quickly recover it)

__Special case of the IFT when $p=1$__

__Theorem:__ $U \subset \mathbb{R}^m$ open, $I=(a,b)$, $F: \begin{array}{l} U\times I \longrightarrow \mathbb{R} \\ (x_1,\dots,x_m,y) \mapsto F(x_1,\dots,x_m,y) \end{array}$ C^1

If $\dfrac{\partial F}{\partial y}(x_0,y_0) \neq 0$ then there exist $r,s>0$ with $B(x_0,r) \subset U$ and $(y_0-s, y_0+s) \subset I$ and $\varphi : B(x_0,r) \to (y_0-s, y_0+s)$ C^1 s.t.

$$\forall (x,y) \in B(x_0,r)\times(y_0-s,y_0+s),\quad F(x,y) = F(x_0,y_0) \Leftrightarrow y = \varphi(x)$$

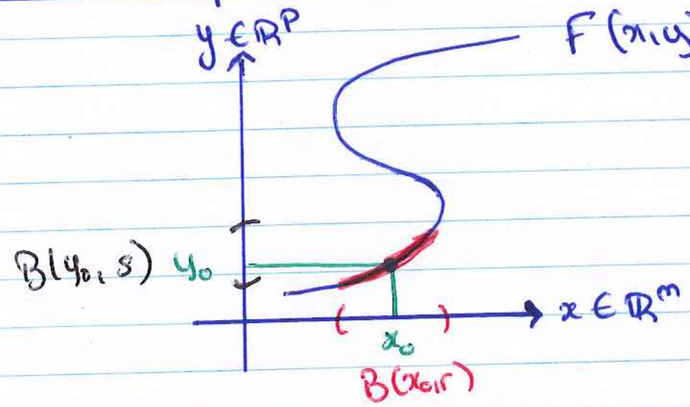
__Remark:__ by computing the $\dfrac{\partial}{\partial x_i}$'s derivative at x_0 of $F(x,\varphi(x)) = F(x_0,y_0)$

we get:

$$\boxed{\frac{\partial \varphi}{\partial x_i}(x_0,y_0) = -\frac{\dfrac{\partial F}{\partial x_i}(x_0,y_0)}{\dfrac{\partial F}{\partial y}(x_0,y_0)}}$$

!

Geometric interpretation:

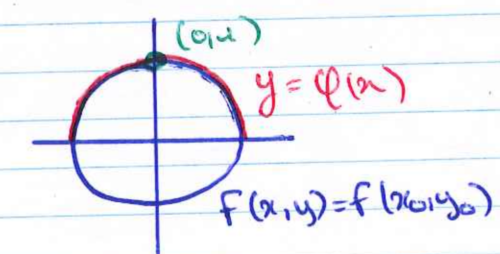


Under the assumptions of the IFT, the level set $F(x,y) = F(x_0,y_0)$ defines locally around (x_0,y_0) a function $y = \varphi(x)$ of class C^1

Example:

$F(x,y) = x^2+y^2$, $(x_0,y_0) = (0,1)$, $\frac{\partial F}{\partial y}(0,1) = 2 \neq 0$

$F(x,y) = F(x_0,y_0) \Leftrightarrow x^2+y^2 = 1$



$$\varphi : \begin{array}{rcl} (-1,1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sqrt{1-x^2} \end{array}$$

$F(x, \varphi(x)) = 1 \Rightarrow x^2 + \varphi(x)^2 = 1 \Rightarrow 2x + 2\varphi(x)\varphi'(x) = 0$

$\Rightarrow 2\varphi(0)\varphi'(0) = 0$

$\Rightarrow \varphi'(0) = 0$

Remark: at $(1,0)$ $\frac{\partial F}{\partial y}(1,0) = 0$ but $\frac{\partial F}{\partial x}(1,0) = 2 \neq 0$

so we can express $F(x,y) = 1$ as a function $x = \varphi(y)$

Homework: Questions from 3.1

The Inverse Function Theorem

⚠ Notice that the domain and codomain have same dimension m

<u>Theorem</u>: $\mathcal{U} \subset \mathbb{R}^m$ open, $f: \mathcal{U} \to \mathbb{R}^m$ C^1, $a \in \mathcal{U}$

If $Df(a)$ is invertible then $\exists\, \mathcal{V}, \mathcal{W} \subset \mathbb{R}^m$ open sets satisfying $a \in \mathcal{V} \subset \mathcal{U}$, $f(a) \in \mathcal{W}$ s.t. $f: \mathcal{V} \to \mathcal{W}$ is a C^1-diffeomorphism.

(ie: $f: \mathcal{V} \to \mathcal{W}$ is C^1, bijective and $f^{-1}: \mathcal{W} \to \mathcal{V}$ is also C^1)

⚠ If $Df(a)$ is invertible then locally around a and $f(a)$ f is a C^1-diffeomorphism : we shrink the domain from $\mathcal{U}$ to a smaller $\mathcal{V}$

<u>Remark</u>: the implicit function theorem and the inverse function theorem are equivalent

Δ <u>ImpFT $\Rightarrow$ InvFT</u>:

Let $f: \mathcal{U} \xrightarrow{C^1} \mathbb{R}^m$, $\mathcal{U} \subset \mathbb{R}^m$ open, $a \in \mathcal{U}$ s.t. $Df(a)$ is invertible.

Define $F: \mathcal{U} \times \mathbb{R}^m \to \mathbb{R}^m$ by $F(x,y) = f(x) - y$

Then $D_x F(a, f(a)) = Df(a)$ is invertible.

[$D_x F$ is invertible so the ImpFT gives x in terms of y]

Hence, by the ImpFT, $\exists\, \varphi: B(f(a), r) \to B(a,s)$ C^1

s.t. $\forall (x,y) \in B(a,s) \times B(f(a), r)$, $F(x,y) = \overbrace{F(a, f(a))}^{0} \Leftrightarrow x = \varphi(y)$

<u>ie</u> $y = f(x) \Leftrightarrow x = \varphi(y)$

so $\varphi = f^{-1}$ for $f: B(a,s) \cap f^{-1}(B(f(a), r)) \longrightarrow B(f(a), r)$

ImvFT $\Rightarrow$ ImpFT:

Let $F: \mathcal{U}\times\mathcal{V} \to \mathbb{R}^p$ C^1, $\mathcal{U}\subset\mathbb{R}^m$ open, $\mathcal{V}\subset\mathbb{R}^p$ open, assume $D_yF(x_0,y_0)$ invertible

Notice that $\mathcal{U}\times\mathcal{V}$ is an open subset of $\mathcal{U}\times\mathcal{V}$ and define

$f: \mathcal{U}\times\mathcal{V} \to \mathbb{R}^{m+p}$ by $f(x,y) = (x, F(x,y))$

$$Df(x_0,y_0) = \begin{pmatrix} I_{m,m} & 0 \\ D_xF(x_0,y_0) & D_yF(x_0,y_0) \end{pmatrix} \text{ is invertible}$$

so, by the ImvFT, we can find $M, N \subset \mathbb{R}^{m+p}$ open s.t.

$(x_0,y_0) \in M \subset \mathcal{U}\times\mathcal{V}$, $(x_0, F(x_0,y_0)) \in N$, $f: M \to N$ C^1-difféo

notice that, by definition of f, $f^{-1}(x, F(x_0,y_0)) = (x, \varphi(x))$

for some φ C^1 defined in a neighborhood of x_0 ↳ we should be a little bit more careful here

fst $\underset{\overset{\shortparallel}{(x,\, F(x,\varphi(x)))}}{f(x,\varphi(x))} = (x, F(x_0,y_0))$, ie $F(x,\varphi(x)) = F(x_0,y_0)$

$\square$

Rem: from $f^{-1}\circ f = id$ we obtain: $Df^{-1}(f(a)) \circ Df(a) = I_{m,m}$

$\Rightarrow Df^{-1}(f(a)) = [Df(a)]^{-1}$

Definitions: • $f: A \to B$ homeomorphism means f bij + f C^0 + f^{-1} C^0

• $\mathcal{U},\mathcal{V} \subset \mathbb{R}^m$ open, $f: \mathcal{U} \to \mathcal{V}$ C^k-diffeomorphism means:

f bijective + f C^k + f^{-1} C^k

Singularity points:

Case 1: Curves in $\mathbb{R}^2$

Observation: A curve may be described in 3 natural ways

① As a graph $S = \{(x,y) \in \mathbb{R}^2 : x \in I,\ y = f(x)\}$

or $S = \{(x,y) \in \mathbb{R}^2 : y \in I,\ x = f(y)\}$

where $f: I \to \mathbb{R}$, $I \subset \mathbb{R}$ open interval

② As a level set: $S = \{(x,y) \in U : F(x,y) = c\}$

where $U \subset \mathbb{R}^2$ is open and $F: U \to \mathbb{R}$ and $c \in \mathbb{R}$

⚠ In this case we can get something too big

$S = \{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$ where $F(x,y) = 0$
$= \mathbb{R}^2$

or too small

$S = \{(x,y) \in \mathbb{R}^2 : \sin^2(x) + \cos^2(y) = 4\} = \emptyset$
$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} = \{(0,0)\}$

③ Parametrically, $S = \{\gamma(t) : t \in I\}$ where $I \subset \mathbb{R}$ open interval
$\gamma: I \to \mathbb{R}^2$

⚠ Again we can get something too small:
$\{\gamma(t) : t \in \mathbb{R}\} = \{(0,0)\}$ for $\gamma(t) = (0,0)$ constant

or too big: type "Peano curve" on google

↳ this last phenomenon is not possible for γ C^1.

Remark 1: A curve represented by a graph may be represented by a level set

Indeed: $\{(x,y)\in\mathbb{R}^2 : x\in I,\ y=f(x)\}$

$= \{(x,y)\in I\times\mathbb{R} : y-f(x)=0\}$

$= \{(x,y)\in I\times\mathbb{R} : F(x,y)=0\}$ where $F(x,y)=y-f(x)$

But the converse is false: the lemniscate

$x^4-x^2+y^2=0$ $\qquad \infty$

may not be described as a graph around $(0,0)$

Remark 2: A curve represented by a graph admits a parametrization

Indeed: $\{(x,y)\in\mathbb{R}^2 : x\in I,\ y=f(x)\}$

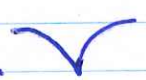
$= \{(t,f(t)) : t\in I\}$

$= \{\gamma(t) : t\in I\}$ where $\gamma(t)=(t,f(t))$

But the converse is false: $\gamma(t)=\left(\frac{t^2-1}{t^2+1},\ \frac{2t(t^2-1)}{(t^2+1)^2}\right)$

gives again the lemniscate ∞ which may not be described as a graph around $(0,0)$

another parametrization is $\gamma(t)=(\sin(t),\sin(t)\cos(t))$

Remark 3: $y^3-x^2=0$.  may be described as the graph of

$f(x)=|x|^{2/3}$ but may not be described as the graph of a C^1 function

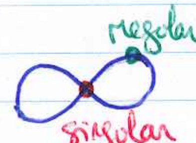
The above observations lead us to the following definitions:

Def. Let $C \subset \mathbb{R}^2$ be a curve. We say that C is regular at $a \in C$ if there exists $\varepsilon > 0$ s.t. $C \cap B(a,\varepsilon)$ is the graph of a C^1 function.
We say that C is singular at a if for every $\varepsilon > 0$, $C \cap B(a,\varepsilon)$ is not the graph of a C^1 function.

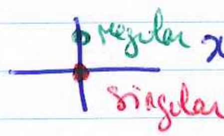
Ex.:  $x^2+y^2=1$ is regular at all its points
is the graph of some $x=g(y)$ C^1
is the graph of some $y=f(x)$ C^1

Ex.: 

$y^3=x^2$

Ex.: 

$x^4-x^2+y^2=0$

Ex.: 

$xy=0$

Theorem. $F: U \to \mathbb{R}$ C^1, $U \subset \mathbb{R}^2$ open, $a = (x_0,y_0) \in U$
If $\nabla F(a) \neq \vec{0}$ then $C = \{(x,y) \in U : F(x,y) = F(x_0,y_0)\}$ is regular at a.

Δ We know that $\nabla F(a) = \left(\frac{\partial F}{\partial x}(a), \frac{\partial F}{\partial y}(a)\right) \neq (0,0)$

Case 1: $\frac{\partial F}{\partial y}(a) \neq 0$ then by the Implicit Function Theorem, then C is locally the graph of a function $y = \varphi(x)$ around a.

Case 2. $\frac{\partial F}{\partial x}(a) \neq 0$ same but $x = \varphi(y)$

$\square$

Theorem: $I \subset \mathbb{R}$ open interval, $\gamma : I \to \mathbb{R}^2$ C^1, $t_0 \in I$

If $\gamma'(t_0) \neq \vec{0}$ then there exists $r > 0$ s.t. $(t_0 - r, t_0 + r) \subset I$ and $C = \{\gamma(t) : t \in (t_0 - r, t_0 + r)\}$ is regular.

Δ $\gamma'(t_0) = (\gamma_1'(t_0), \gamma_2'(t_0))$ so WLOG we may assume that $\gamma_1'(t_0) \neq 0$.

By the Inverse Function Theorem, $\exists r > 0$ s.t.

$J = (t_0 - r, t_0 + r) \subset I$, $K = \gamma_1(J)$ open interval, $\gamma_1 : J \to K$ bijection and $\gamma_1^{-1} : K \to J$ is C^1.

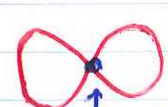
Define $f : K \to \mathbb{R}$ by $f(x) = \gamma_2(\gamma_1^{-1}(x))$

then
$$\{(x,y) \in \mathbb{R}^2 : x \in K, \ y = f(x)\}$$
$$= \{(x,y) \in \mathbb{R}^2 : t \in J, \ x = \gamma_1(t), \ y = \gamma_2(t)\}$$
$$= \{\gamma(t) : t \in J\}$$

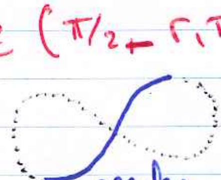
The idea is the following: amount to " $x \underset{\text{bij}}{\overset{\gamma_1}{\longleftrightarrow}} t \xrightarrow{\gamma_2} y$ " $\square$

⚠ Beware: the domain of f is shrunk, we don't conclude that $\{f(t) : t \in I\}$ is regular only $\{f(t) : \text{for } t \text{ close to } t_0\}$

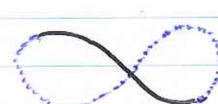
Ex: $\gamma(t) = (\sin(t), \sin(t)\cos(t))$

$t \in \mathbb{R}$: 

singular at $(0,0)$

$t \in (\pi/2 - r, \pi/2 + r)$ 

regular

$t \in (3\pi/2 - r, 3\pi/2 + r)$ 

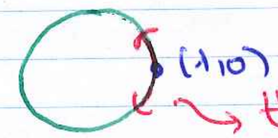
regular

Case 2: The general case

Def: Let $M \subset \mathbb{R}^N$, $a \in M$. We say that M is regular of dimension d at a if there exists $\varepsilon > 0$ s.t., up to permuting the variables, $B(a,\varepsilon) \cap M$ is the graph of a C^1-function $f : \mathcal{U} \to \mathbb{R}^{N-d}$ where $\mathcal{U} \subset \mathbb{R}^d$ is open

⚠ "Up to permuting the variables" means that we express $N-d$ variables in terms of d variables but not necessarily the $N-d$ last in terms of the d first.

Ex: $M = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ is regular of dim 1 at $(1,0)$



this part may not be described as a graph $y = \varphi(x)$ but it can be described as a graph $x = \varphi(y)$

Theorem: $\mathcal{U} \subset \mathbb{R}^N$ open, $F : \mathcal{U} \to \mathbb{R}^{N-d}$ C^1, $a \in \mathcal{U}$.
If $\operatorname{rank}(DF(a)) = N-d$ then $M = \{x \in \mathcal{U} : F(x) = F(a)\}$ is regular of dimension d at a.

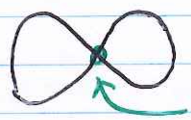
Δ Let's denote $DF(a) = (v_1\ v_2\ \cdots\ v_N) \in M_{N-d,N}(\mathbb{R})$, by assumption we may find $i_1, \dots, i_{N-d}$ s.t. $v_{i_1}, \dots, v_{i_{N-d}}$ are linearly independant then $D_{(x_{i_1}, \dots, x_{i_{N-d}})} F(a)$ is invertible.
By the ImpFT, we may locally describe M around a as a C^1 graph $(x_{i_1}, \dots, x_{i_{N-d}}) = \varphi(\text{of the } d \text{ remaining variables})$ $\square$

<u>Theorem</u>: $\mathcal{U} \subset \mathbb{R}^d$ open, $\sigma : \mathcal{U} \to \mathbb{R}^N$ C^1, $t_0 \in \mathcal{U}$.

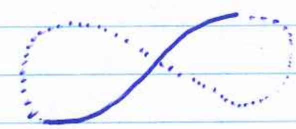
If $\operatorname{rank}(D\sigma(t_0)) = d$ then $\exists r > 0$ s.t. $B(t_0, r) \subset \mathcal{U}$ and

$M = \{\sigma(t) : t \in B(t_0, r)\}$ is regular of dimension d

⚠ Again, as in the planar curve case, we shrunk the domain to avoid "self intersection"



$t \in \mathbb{R}$
not regular



$t \in (-1,1)$
regular

Δ

$$D\sigma(t_0) = \begin{array}{c} \\ N \updownarrow \end{array} \underbrace{\begin{pmatrix} \nabla\sigma_1(t_0) \\ \nabla\sigma_2(t_0) \\ \vdots \\ \nabla\sigma_N(t_0) \end{pmatrix}}_{d}$$ is of rank d, up to permuting

the components we may assume that $\nabla\sigma_1(t_0), \dots, \nabla\sigma_d(t_0)$ are linearly independant

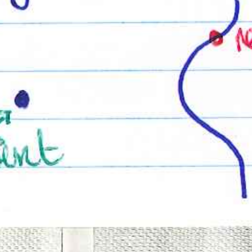
hence, by the Inv FT, $\varphi : t \mapsto (\sigma_1(t), \dots, \sigma_d(t))$ is a C^1-diffeo for $t_0 \in v$ small enough ($v \longrightarrow w$)

We define $f : w \to \mathbb{R}^{N-d}$ by $f(\underbrace{x_1, \dots, x_d}_{x}) = (\sigma_{d+1}(\varphi^{-1}(x)), \dots, \sigma_N(\varphi^{-1}(x)))$

then $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^{m-d}, x \in w, y = f(x)\} = \{\sigma(t) : t \in v\}$ □

<u>Def</u>. $M \subset \mathbb{R}^N$, $a \in M$. We say that a is a singular point of M if a is not a regular point of "maximal dimension"
Otherwise, we say that a is a regular point of M (with no precision about the dimension)

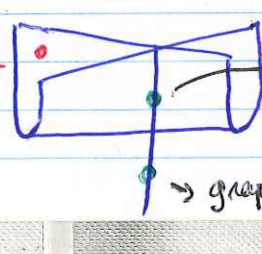
<u>Ex</u>: $x^2 + y^2 - x^3 = 0$ (it's a curve, dim 1)



regular point

singular point
(dim 0)

<u>Ex</u>: Whitney's umbrella $x^2 - zy^2 = 0$



regular

→ not a graph

• singular

→ graph of dim $1 < 2$

⚠

You may be tempted to use the following $\boxed{\text{false}}$ statement: **DO NOT**, it is $\boxed{\text{false}}$ ⚠

FALSE STATEMENT

Let $\sigma : \mathcal{U} \xrightarrow{C^1} \mathbb{R}^N$ be a parametrization where $\mathcal{U} \subset \mathbb{R}^d$ open

If ① $\forall t \in \mathcal{U}$, $D\sigma(t)$ is of rank d

② $t \neq t' \Rightarrow \sigma(t) \neq \sigma(t')$

then $\underset{\mathbb{R}^N}{\underset{\cap}{C}} = \{\sigma(t) : t \in \mathcal{U}\}$ is regular of dimension d

BAD PROOF

Δ Indeed, by 1, $\forall t_0 \in \mathcal{U}$, $\exists \varepsilon > 0$, s.t. $\{\sigma(t) : t \in (t_0 - \varepsilon, t_0 + \varepsilon)\}$ is regular (ie there is no cusp)

and by 2 there is no self intersection, so we can't have something like ∞ $\{(\sin(t), \sin(2t)) : t \in \mathbb{R}\}$ $\square$

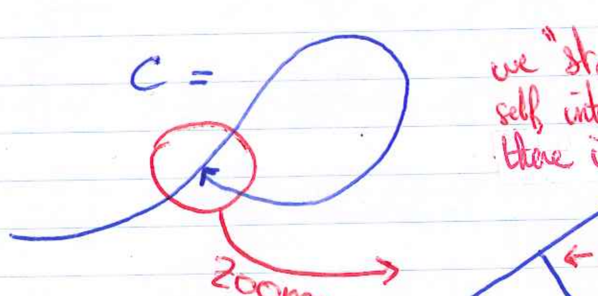
<u>Comment</u>: the last part of the proof is false:

take $\sigma(t) = (\sin(t), \sin(2t))$, $t \in (-1, \pi)$

then σ satisfies the assumptions ① and ② but

$C = \{\sigma(t) : t \in (-1, \pi)\}$ is not regular at $(0,0)$.

counter-example

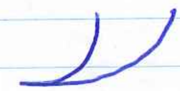


we "stop" the parametrization just before self intersecting so σ is 1-to-1 but there is no "hole" between the two branches

← can't be a graph

Ex: a curve in the 3-dimensional Euclidean space

- $t \mapsto (t^3, t^2, t^6)$ parametrization
- $\begin{cases} x^2 - z = 0 \\ y^3 - z = 0 \end{cases}$ level set / implicit equation



both define the same curve $C \subset \mathbb{R}^3$

Define $F(x,y,z) = (x^2 - z, y^3 - z)$ then $C = \{(x,y,z) : F(x,y,z) = (0,0)\}$

$DF(x,y,z) = \begin{pmatrix} 2x & 0 & -1 \\ 0 & 3y^2 & -1 \end{pmatrix}$ is of rank $2 \overset{3-1}{=}$ except at $(0,0,0) \in C$

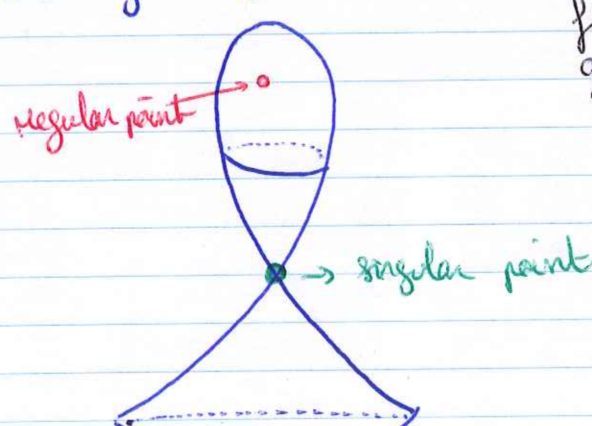
By the above theorem, C is regular for $(x,y,z) \neq (0,0,0)$

Graphically, we see that C is singular at $(0,0,0)$ → Prove it rigorously!

Ex: $x^2 + y^2 - z^2(1-z) = 0$

Rem: this surface is easy to draw, for $z = z_0$ fixed, the distance ρ at the O_z axis is given by $\rho^2 = z_0^2(1 - z_0)$

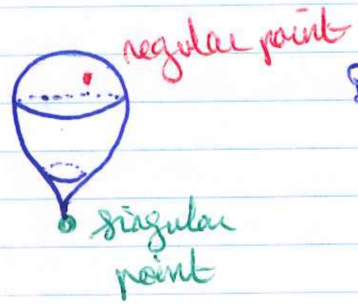
$z \mapsto \sqrt{z^2(1-z)}$



(Indeed, if we remove this point we have to path-connected components whereas if we remove a point of a graph $B(0,r) \subset \mathbb{R}^2 \to \mathbb{R}$, we have one)

$$DF(x,y,z) = (2x \;\; 2y \;\; z(3z-2))$$

Ex: $x^2 + y^2 - z^3(1-z) = 0$



$$DF(x,y,z) = (2x \;\; 2y \;\; z^2(-3+4z))$$

Exercise : $C = \{ (x,y) \in \mathbb{R}^2 : x^3+y^3-3xy=0 \}$ "Folium of Descartes"

1. Study the singular points of C
2. Find a parametric description of C
   (Hint: study $C \cap \{ y = tx \}$
   and use it to draw C

Exercise: Prove that the following sets are singular at the origine

1) $M = \{ (x,y) \in \mathbb{R}^2 : y = |x| \}$
2) $M = \{ (x,y) \in \mathbb{R}^2 : x^2 = y^2 \}$
3) $M = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2 \}$
4) $M = \{ (x,y) \in \mathbb{R}^2 : x^2 = y^3 \}$

Exercise: Define $C \subset \mathbb{R}^3$ implicitely by

$$\begin{cases} x^2+y^2+z^2 = R^2 \\ x^2+y^2-2x=0 \end{cases}$$ where $R>0$ is fixed

1. Prove that C is regular for $R \neq 2$
2. What do we get for $R=2$

I wrote these solutions quickly after class
$\rightarrow$ DOUBLE CHECK EVERYTHING

Solutions:

Exo 1. $C \cap \{y = tx\} \rightsquigarrow \begin{cases} y = tx \\ x^3 + y^3 - 3xy = 0 \end{cases}$

$$\Rightarrow \begin{cases} y = tx \\ x^3 + t^3 x^3 - 3t\, x^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = tx \\ x^2\left((1+t^3)x - 3t\right) = 0 \end{cases}$$

$$\Rightarrow x,y = (0,0) \text{ or } (x,y) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$$

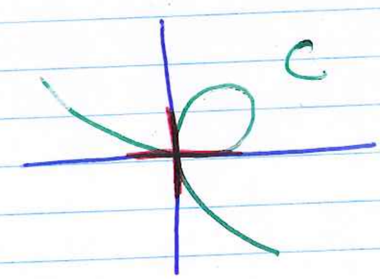
we study $x(t) = \frac{3t}{1+t^3}$ and $y(t) = \frac{3t^2}{1+t^3}$

t	$-\infty$		-1		0		$2^{-1/3}$		$2^{1/3}$		$+\infty$
$x(t)$	0	$\nearrow$	$+\infty \,\|\, -\infty$	$\nearrow$	0	$\nearrow$	$2^{2/3}$	$\searrow$	$2^{1/3}$	$\searrow$	0
$y(t)$	0	$\searrow$	$-\infty \,\|\, +\infty$	$\searrow$	0	$\nearrow$	$2^{1/3}$	$\nearrow$	$2^{2/3}$	$\searrow$	0

$\frac{y}{x} = t \to -\infty$ vertical tangent at $(0,0)$

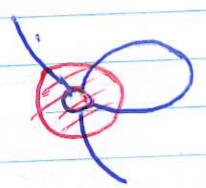
$\frac{y}{x} = t \to 0$ horizontal tangent at $(0,0)$

$\frac{y}{x} = t \to +\infty$ vertical tangent line at $(0,0)$



$f(x,y) = x^3 + y^3 - 3xy \qquad Df(x,y) = \begin{pmatrix} 3x^2 - 3y & 3y^2 - 3x \end{pmatrix}$

is of max rank on $C \setminus \{0\}$ so $C \setminus \{0\}$ is regular

At $(0,0)$:  $(C \setminus \{0\}) \cap B(0,\varepsilon)$ has 4 path-connected components : C is not regular of dim 1 at $(0,0)$

IDEM: Don't trust me!

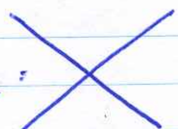
Exo 2

① • Assume $y = \varphi(x)$ in a $B(0,\varepsilon)$

then $\varphi(x) = |x|$ is not C^1

• Assume $x = \varphi(y)$ in a $B(0,\varepsilon)$

then $|\varphi(y)| = y \Rightarrow \varphi(y) = \pm y$: not a graph (2 possibles x-values for a y

② 

$C: x^2 - y^2 = (x-y)(x+y)$

Method 1: $y = \varphi(x) \Rightarrow (x - \varphi(x))(x + \varphi(x)) = 0$

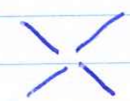
$\Rightarrow \varphi(x) = \pm x$ ⇝ not a graph

$x = \varphi(y) \Rightarrow \varphi(y) = \pm y$ ⇝ not a graph

Method 2: if $\{(x, \varphi(x))\} = B(0,\varepsilon) \cap$ for $\varphi : I \to \mathbb{R}$ ⇝ small interval

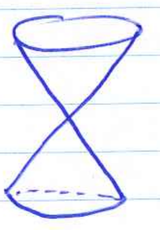
then $M \cap B(0,\varepsilon) = F(I)$ for $F(x) = (x, \varphi(x))$

$\Rightarrow (M \cap B(0,\varepsilon)) \setminus \{0\} = F(I \setminus \{0\})$

 $= F(\longleftrightarrow \cup \longleftrightarrow)$ contradiction

↳ 4 components ↳ 2 components

↳ at most 2 components since the C^0 image of a p-c is p-c

③ 

Method 1: $x = \varphi(y,z) \Rightarrow \varphi(y,z) = \pm\sqrt{z^2 - y^2}$ not a graph

$y = \varphi(x,z) \Rightarrow \varphi(x,z) = \pm\sqrt{z^2 - x^2}$ not a graph

$z = \varphi(x,y) \Rightarrow \varphi(x,y) = \pm\sqrt{x^2 + y^2}$

Method 2: if M is regular then $(B(0,\varepsilon) \cap M) \setminus \{0\}$ ← 2 comp

is

$B(0,\varepsilon) \setminus \{0\} \subset \mathbb{R}^2$ ← 1 comp

do as above

④ $x = \varphi(y) \Rightarrow \varphi(y)^2 = y^3 \Rightarrow \varphi(y) = \pm y^{3/2}$ not a graph

$y = \varphi(x) \Rightarrow x^2 = \varphi(x)^3 \Rightarrow \varphi(x) = x^{2/3}$ not C^1

Why are we so interested by sets that are locally a graph?

Extra: (not part of MAT 237)

Assume that $M \subset \mathbb{R}^N$ is regular of dimension d at $a \in M$, then we may locally flatten M around a: around a M looks like $\mathbb{R}^d$.

Formally: $\exists \mathcal{U} \subset \mathbb{R}^N$ an open subset containing a
$\exists \mathcal{V} \subset \mathbb{R}^N$ an open subset containing $\vec{0}$
$\exists F : \mathcal{U} \to \mathcal{V}$ a C^1-diffeomorphism (φ is bijective, φ and φ^{-1} are C^1)

such that $F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d})$

Δ By definition $\exists \varepsilon > 0$ s.t. $B(a, \varepsilon) \cap M = \{(x, \varphi(x)) : x \in \mathcal{W}\}$ where $\mathcal{W} \subset \mathbb{R}^d$ is open and $\varphi : \mathcal{W} \to \mathbb{R}^{N-d}$. (up to permuting the coordinates)

Define $F : \mathcal{W} \times \mathbb{R}^{N-d} \to \mathbb{R}^N$ by

$$F(x_1, \dots, x_d, y_1, \dots, y_{N-d}) = (x_1 - a_1, \dots, x_d - a_d, y_1 - \varphi_1(x), \dots, y_{N-d} - \varphi_{N-d}(x))$$

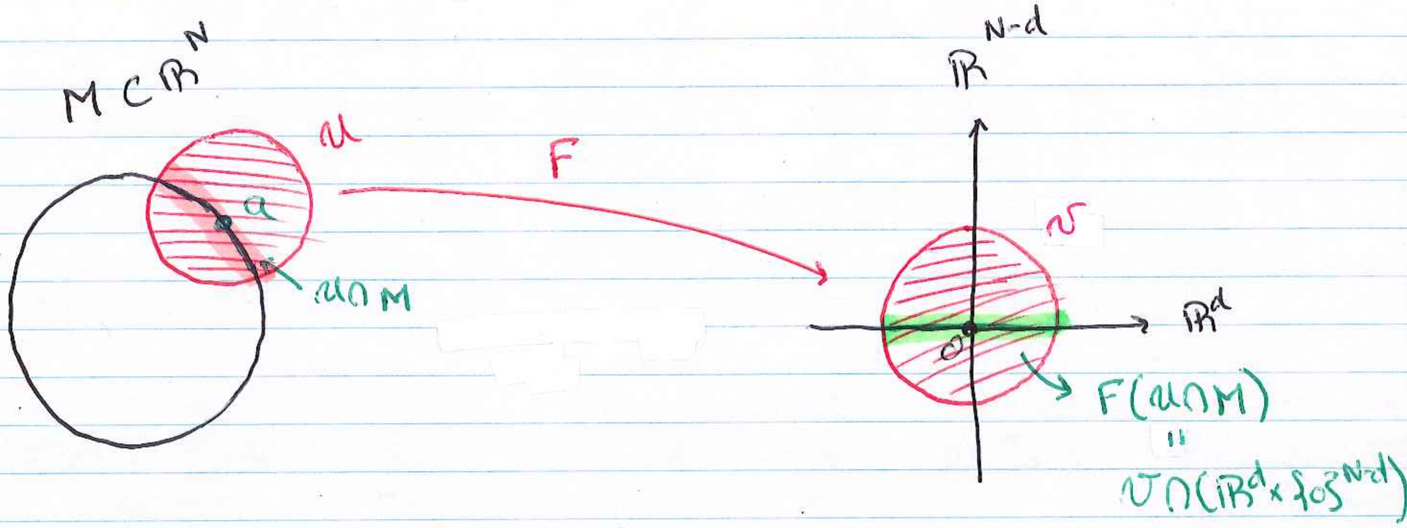
Then $DF(a) = \begin{pmatrix} I_{d,d} & 0 \\ * & I_{N-d,N-d} \end{pmatrix}$ is invertible, so by the inverse function theorem $\exists \mathcal{U} \subset \mathbb{R}^N$ open containing a, $\exists \mathcal{V} \subset \mathbb{R}^N$ open containing $F(a) = 0$ s.t. $F : \mathcal{U} \to \mathcal{V}$ is a C^1-diffeomorphism.

Moreover, $F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d})$ by definition of F $\square$

We say that M is a d-dim C^1-submanifold if it is everywhere regular of dim d.

Since "being C^1" is a local property, it allows to define C^1-functions defined on M

Ex :



So if we have a function $f : M \longrightarrow \mathbb{R}$, we may study

$$\tilde{f} = f \circ (F^{-1}) : F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d}) \longrightarrow \mathbb{R}$$

then we may see $\tilde{f}$ as a function defined on $\mathbb{R}^d$ (locally) and do calculus ...

$\square$ End of extra.

⚠ don't forget this assumption

Transformations / Change of coordinates

<u>Theorem</u>: $\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}^n$ C^1 and injective

The following are equivalent:

(i) $\forall x_0 \in \Omega$, $Df(x_0)$ is invertible

(ii) $f(\Omega)$ is open and $f : \Omega \to f(\Omega)$ is a C^1-diffeomorphism

Δ <u>$i \Rightarrow ii$</u>: $f : \Omega \to f(\Omega)$ is obviously a bijection

Let $y_0 = f(x_0) \in f(\Omega)$. Since $Df(x_0)$ is invertible, by the inverse function theorem $\exists M, N \subset \mathbb{R}^n$ open with $x_0 \in M$, $y_0 \in N$ s.t. $f : M \to N$ is a C^1-diffeo.

Then $y_0 \in N = f(M) \subset f(\Omega)$ with N open so $\exists r > 0$ s.t. $B(y_0, r) \subset N \subset f(\Omega)$

Since it is true for any $y_0 \in f(\Omega)$, $f(\Omega)$ is open

Moreover since $f^{-1} : N \to M$ is C^1, f^{-1} is C^1 at y_0

<u>$ii \Rightarrow i$</u>: we have $f^{-1} \circ f = id$ so $D(f^{-1})(f(x_0)) \cdot Df(x_0) = I_{n,n}$ and $Df(x_0)$ is invertible

□

⚠ The injective assumption is important: $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (e^x \cos y, e^x \sin y)$ satisfies $\forall (x_0, y_0) \in \mathbb{R}^2$ $Df(x_0, y_0)$ is invertible but is not injective.

Exercise 1: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (e^x \cos y, e^x \sin y)$

Set $\mathcal{U} = \{(x,y) \in \mathbb{R}^2 : y \in (0, 2\pi)\}$

① Compute $f(\mathcal{U})$

② Prove that $f(\mathcal{U})$ is open and $f: \mathcal{U} \to f(\mathcal{U})$ is a C^1-diffeomorphism

③ If $g = f^{-1}: f(\mathcal{U}) \to \mathcal{U}$, compute $Dg(0,1)$

Exercise 2: let $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (e^{2y} + e^{2z}, e^{2x} - e^{2z}, x-y)$

Prove that $f(\mathbb{R}^3) \subsetneq \mathbb{R}^3$

and that $f(\mathbb{R}^3)$ is open

C^1-diffeomorphisms are important because they allow to study a C^1-function after a change of coordinates

Indeed, let $f : \mathcal{U} \to \mathbb{R}^p$ be C^1 with $\mathcal{U} \subset \mathbb{R}^n$ open and let $\varphi : \mathcal{V} \to \mathcal{U}$ be a C^1-diffeomorphism.
Then, if we set $\tilde{f} = f \circ \varphi : \mathcal{V} \to \mathbb{R}^p$ we have

$$\begin{cases} \tilde{f} = f \circ \varphi \\ f = \tilde{f} \circ \varphi^{-1} \end{cases}$$

so we may either study f or $\tilde{f}$.

<u>Ex</u>: polar coordinates

$$\mathcal{U} = \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}, \quad \mathcal{V} = \{(r,\theta) : r > 0, |\theta| < \pi\}$$

then $\varphi : \mathcal{V} \to \mathcal{U}$ defined by $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$ is a C^1-diffeomorphism.

Hence, instead of working with $f : \begin{array}{l} \mathcal{U} \to \mathbb{R}^p \\ (x,y) \mapsto f(x,y) \end{array}$
we may work with $\tilde{f} : \mathcal{V} \to \mathbb{R}^p$ defined by
$\tilde{f}(r,\theta) = f(\varphi(r,\theta))$ (which may be useful if f is invariant w.r.t rotation centered at 0)

It's common to simply write $f(r,\theta)$ instead of $f(\varphi(r,\theta))$ but be careful, that's an abuse of notation