

## High School Vectors Test Answers and Solutions:

### Answers to the Test:

1. T 2. F 3. T 4. F 5. T 6. T 7. C 8. A 9. B 10. C

### Solutions and Comments:

1.  $\left[ \frac{2}{3} \quad -\frac{2}{3} \quad \frac{1}{3} \right]$  is a unit vector. This is True, since the length of the vector is 1:

$$\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{\frac{9}{9}} = 1.$$

2. The vectors  $[ 2 \ 4 \ 8 ]$  and  $[ 4 \ 7 \ -4 ]$  are perpendicular. This is False, since the dot product of the two vectors is not zero:

$$(2)(4) + (4)(7) + (8)(-4) = 8 + 28 - 32 = -4 \neq 0.$$

3. If the vector  $[ a \ b \ c ]$  is perpendicular to both of the vectors  $[ 1 \ -1 \ 1 ]$  and  $[ 1 \ 0 \ -1 ]$  then  $b = 2a$ . This is True. If the vector  $[ a \ b \ c ]$  is perpendicular to both of the other two vectors, then both dot products must be zero:

$$a - c = 0 \text{ and } a - b + c = 0 \Rightarrow a = c \text{ and } b = a + c \Rightarrow b = 2a.$$

4. The angle between the two vectors  $[ 2 \ -3 \ 1 ]$  and  $[ 1 \ 6 \ 2 ]$  is in the third quadrant. This is False, obviously, since the angle between two vectors is defined to be an angle between 0 and  $\pi$ , so the angle between two vectors is always in the first or second quadrant, never in the third quadrant. If you do the calculations:

$$\cos \theta = \frac{(2)(1) + (-3)(6) + (1)(2)}{\sqrt{4 + 9 + 1}\sqrt{1 + 36 + 4}} = -\frac{14}{\sqrt{14}\sqrt{41}} < 0 \Rightarrow \frac{\pi}{2} < \theta < \pi.$$

5. The equation of the plane passing through the three points  $(1, 3, 3)$ ,  $(2, 3, 5)$  and  $(1, 1, 1)$  is  $2x + y - z = 2$ . This is True. The easiest way to confirm it is to check that all three (non-collinear) points satisfy the equation, which they do!
6. The points with coordinates  $(1, 2, -3)$ ,  $(1, -2, -3)$  and  $(1, 2, 0)$  are the vertices of a right angled triangle. This is True.

Consider the vertex  $(1, 2, -3)$  and let  $\alpha$  be the interior angle of the triangle at this vertex, between the two vectors  $[ 1 - 1 \quad -2 - 2 \quad -3 + 3 ] = [ 0 \quad -4 \quad 0 ]$  and  $[ 1 - 1 \quad 2 - 2 \quad 0 + 3 ] = [ 0 \quad 0 \quad 3 ]$ . Then

$$\cos \alpha = \frac{(0)(0) + (-4)(0) + (0)(3)}{\sqrt{0^2 + (-4)^2 + 0^2}\sqrt{0^2 + 0^2 + 3^2}} = 0 \Rightarrow \alpha = \frac{\pi}{2}.$$

7. Suppose the angle between the two non-zero vectors  $\vec{u}$  and  $\vec{v}$  is  $\theta$ . Since

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

and both  $\vec{u}$  and  $\vec{v}$  are non-zero, C is false. Aside: the other three options are all true formulas about vectors that you should be familiar with.

8. At the intersection of the two lines

$$\begin{cases} 1+t = u \\ -t = 5-3u \\ 2+t = 1+u \end{cases} \Rightarrow \begin{cases} 1 = 5-2u \\ 2 = 6-2u \end{cases} \Rightarrow u = 2, t = 1.$$

So the intersection point is  $(2, -1, 3)$ .

9. To find the intersection point of the line with parametric equations

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 1+t & 4t & 2+t \end{bmatrix}$$

and the plane with equation  $4x - y + 2z = 4$ , substitute for  $x, y, z$  from the line into the equation of the plane, and solve for  $t$ :

$$4(1+t) - 4t + 2(2+t) = 4 \Leftrightarrow 8 + 2t = 4 \Leftrightarrow t = -2.$$

So the point of intersection of the line and the plane is  $(x, y, z) = (-1, -8, 0)$ .

10. The minimum distance from the point with coordinates  $(1, -1, 1)$  to the plane with equation  $x + y + z = 4$  is  $\sqrt{3}$ . There are two ways to show this.

**Use Formula:** the minimum distance from a point  $(p, q, r)$  to the plane with equation  $ax + by + cz = d$  is given by

$$D = \frac{|ap + bq + cr - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

We have  $(a, b, c) = (1, 1, 1)$ ,  $(p, q, r) = (1, -1, 1)$  and  $d = 4$ , so

$$D = \frac{|(1)(1) + (1)(-1) + (1)(1) - 4|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{|-3|}{\sqrt{3}} = \sqrt{3}.$$

**Find The Closest Point:** the normal to the plane is  $\vec{n} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ ; so the parametric equations of the line normal to the plane and passing through the point  $(1, -1, 1)$  are given by

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 1+t & -1+t & 1+t \end{bmatrix}.$$

Find the point of intersection of this line with the plane, as in Question 9.; it is  $(2, 0, 2)$ . So the minimum distance from the point  $(1, -1, 1)$  to the plane is

$$D = \sqrt{(2-1)^2 + (0+1)^2 + (2-1)^2} = \sqrt{1+1+1} = \sqrt{3}.$$