MAT335H1F Lec0101 Burbulla

Chapter 9 and 10 Lecture Notes

Fall 2012

Chapter 9: Symbolic Dynamics
Chapter 10: Chaos

Chapter 9: Symbolic Dynamics
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Points with Prime Period 3 for $Q_c(x) = x^2 + c$

Suppose we try to find exactly what are the points with prime period 3 for $Q_c(x) = x^2 + c$.

$$Q_c^3(x) = x \Leftrightarrow \left((x^2+c)^2+c\right)^2 + c = x \Leftrightarrow x^2 - x + c = 0$$

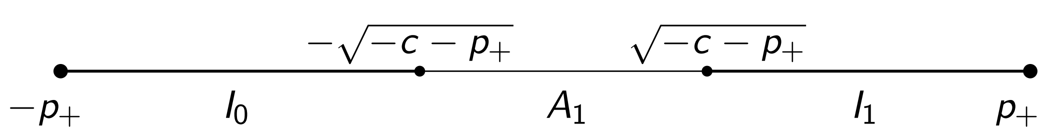
or x is a root of the sixth degree polynomial:

$$x^6 + x^5 + (3c+1)x^4 + (2c+1)x^3 + (3c^2+3c+1)x^2 + (c^2+2c+1)x + c^3 + 2c^2 + c + 1.$$

This is not easy to solve! What is needed to describe theoretically the dynamics of the quadratic family is another way to look at the whole situation. Analytic calculations are just too difficult. Enter symbolic dynamics! That is, abstract the situation in some fashion such that the calculations become easier!

Recall (or Check) the Following for Q_c, $c < -2$

1. Fixed points of Q_c are $p_- = \dfrac{1-\sqrt{1-4c}}{2}$, $p_+ = \dfrac{1+\sqrt{1-4c}}{2}$
2. $I = [-p_+, p_+]$
3. $x_0 \notin I \Rightarrow x_n \to \infty$
4. $\Lambda = \{x \in I \mid Q_c^n(x) \in I \text{ for all } n\}$
5. $A_1 = (-\sqrt{-c-p_+}, \sqrt{-c-p_+})$ is the open subinterval of I that contains all the points x_0 such that $x_1 < -p_+$; i.e. the orbit of x_0 under Q_c 'escapes to infinity' after one iteration.
6. A_1 divides I into two disjoint closed subintervals, I_0 and I_1:



Itineraries

Now let $x_0 \in \Lambda \subset I_0 \cup I_1$. For all $n, x_n = Q_c^n(x_0) \in I_0 \cup I_1$.
Definition: The itinerary of x_0 is the sequence $S(x_0)$ of 0's and 1's given by

$$S(x_0) = (s_0 s_1 s_2 s_3 \dots s_n \dots)$$

such that

$$s_n = \begin{cases} 0 & \text{if} \quad x_n \in I_0 \\ 1 & \text{if} \quad x_n \in I_1 \end{cases}.$$

The itinerary of x_0 is a simplified, symbolic representation of the orbit of x_0 under Q_c.

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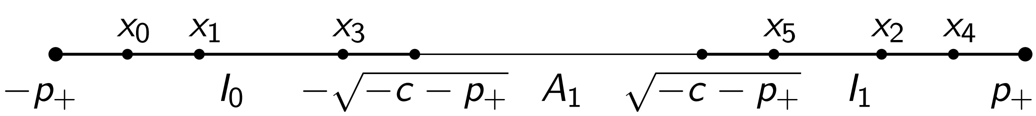
Examples

Example 1: $S(p_+) = (1111\dots)$ since $x_n = p_+ \in I_1$, for all n.
Example 2: $S(-p_+) = (0111\dots)$ since $x_0 \in I_0$ but $x_n = p_+ \in I_1$, for all $n > 0$.
Example 3: $S(p_-) = (0000\dots)$ since $x_n = p_- \in I_0$, for all n.
Example 4: If the first six terms in the orbit of x_0 under Q_c look like



then

$$S(x_0) = (001011\dots)$$

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What Is Sequence Space?

Definition 1: Sequence space is the set

$$\Sigma = \{(s_0 s_1 s_2 \ldots s_n \ldots) \mid s_i = 0 \text{ or } 1, \text{ for } i = 0, 1, 2, \ldots, n, \ldots\}.$$

Every itinerary of every possible choice of $x_0 \in \Lambda$ is in Σ.
Definition 2: The distance between two points $\mathbf{s} = (s_0 s_1 s_2 \ldots)$ and $\mathbf{t} = (t_0 t_1 t_2 \ldots)$ in Σ is defined to be

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

This infinite series always converges since $|s_i - t_i| = 0$ or 1; thus

$$d[\mathbf{s}, \mathbf{t}] \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - 1/2} = 2.$$

Examples

Let $\mathbf{s} = (000000\ldots), \mathbf{t} = (111111\ldots), \mathbf{u} = (010101\ldots)$; then

1. $$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - 1/2} = 2.$$

2. $$d[\mathbf{t}, \mathbf{u}] = 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots = \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{1 - 1/4} = \frac{4}{3}.$$

3. $$d[\mathbf{u}, \mathbf{s}] = \frac{1}{2} + \frac{1}{2^3} + \cdots = \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2} \frac{1}{1 - 1/4} = \frac{2}{3}.$$

d Is Called a Distance Function, or Metric, on Σ

$d : \Sigma \longrightarrow \mathbb{R}$ satisfies the following three conditions. For every $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \Sigma$:

1. Non-negativity: $d[\mathbf{s}, \mathbf{t}] \geq 0$, and $d[\mathbf{s}, \mathbf{t}] = 0 \Leftrightarrow \mathbf{s} = \mathbf{t}$.
2. Symmetry: $d[\mathbf{s}, \mathbf{t}] = d[\mathbf{t}, \mathbf{s}]$
3. Triangle Inequality: $d[\mathbf{s}, \mathbf{u}] \leq d[\mathbf{s}, \mathbf{t}] + d[\mathbf{t}, \mathbf{u}]$

Σ along with d is called a metric space. It is a space in which we can talk about distances, but it is not like the familiar spaces $\mathbb{R}$ or $\mathbb{R}^2$. For instance, the farthest apart any two points can be in Σ is 2. This is not at all like $\mathbb{R}$ in which you could choose two numbers on the real line as far apart as you like. Nevertheless, with d defined on Σ as above we can think geometrically in Σ.

Proof of the Three Properties

The proofs of the three properties of d depend on the analogous properties of absolute value, which is the function used in $\mathbb{R}$ to define the distance between two real numbers a and b, $|a - b|$.

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \geq 0; d[\mathbf{s}, \mathbf{t}] = 0 \Leftrightarrow |s_i - t_i| = 0 \Leftrightarrow s_i = t_i \Leftrightarrow \mathbf{s} = \mathbf{t}$$

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=0}^{\infty} \frac{|t_i - s_i|}{2^i} = d[\mathbf{t}, \mathbf{s}]$$

$$d[\mathbf{s}, \mathbf{u}] = \sum_{i=0}^{\infty} \frac{|s_i - u_i|}{2^i} = \sum_{i=0}^{\infty} \frac{|s_i - t_i + t_i - u_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{|s_i - t_i| + |t_i - u_i|}{2^i}$$

$$= d[\mathbf{s}, \mathbf{t}] + d[\mathbf{t}, \mathbf{u}]$$

The Proximity Theorem

With the above metric d defined on Σ we can determine when two sequences in sequence space are 'close together'. In fact, it turns out to be easy to tell if two sequences are close together:

The Proximity Theorem: *Let* $\mathbf{s}, \mathbf{t} \in \Sigma$ *such that* $s_i = t_i$ *for* $0 \leq i \leq n$. *Then*

$$d[\mathbf{s}, \mathbf{t}] \leq \frac{1}{2^n}.$$

Conversely, if

$$d[\mathbf{s}, \mathbf{t}] < \frac{1}{2^n},$$

then $s_i = t_i$ *for* $0 \leq i \leq n$.

Ie. two sequences are close together if their first few entries agree.

Proof of the Proximity Theorem

Suppose $s_i = t_i$ for $0 \leq i \leq n$, then

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \frac{1}{1 - 1/2} = \frac{1}{2^n}$$

On the other hand, if $s_j \neq t_j$, for some $j \leq n$, then

$$d[\mathbf{s}, \mathbf{t}] \geq \frac{1}{2^j} \geq \frac{1}{2^n}.$$

Consequently, if

$$d[\mathbf{s}, \mathbf{t}] < \frac{1}{2^n},$$

then $s_i = t_i$ for all $0 \leq i \leq n$.

Calculus Review: Limits and Continuity

Recall, from MAT137H1Y, the following definitions:

1. $\lim_{x \to a} f(x) = L$ if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.
2. f is continuous at the point c if $\lim_{x \to c} f(x) = f(c)$
3. f is continuous on the open interval (a, b) if $\lim_{x \to c} f(x) = f(c)$ for all $c \in (a, b)$.
4. f is continuous on the closed interval $[a, b]$ if $\lim_{x \to c} f(x) = f(c)$ for all $c \in (a, b)$, and
$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$
5. $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on $\mathbb{R}$ if it is continuous at every point $c \in \mathbb{R}$.

Four Definitions, and New Ways to Describe Continuity

Definition 1: A subset $A \subset \mathbb{R}$ is open if it is a union of open intervals.
Definition 2: A subset $A \subset \mathbb{R}$ is closed if it is the complement of an open set. That is, $\mathbb{R} - A$ is open.
Definition 3: The closure of a subset $A \subset \mathbb{R}$ is $\overline{A}$, the intersection of all closed sets that contain A.
Definition 4: If Y is a subset of $\mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$, then $f^{-1}(Y) = \{x \in \mathbb{R} \mid f(x) \in Y\}$.
Theorem: Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function. The following four statements are equivalent:

1. f is continuous on $\mathbb{R}$.
2. $f^{-1}(A)$ is an open set for any open set $A \subset \mathbb{R}$.
3. For every subset $A \subset \mathbb{R}$, $f\left(\overline{A}\right) \subset \overline{f(A)}$.
4. $f^{-1}(A)$ is a closed set for any closed set $A \subset \mathbb{R}$.

Examples and Comments

1. $\mathbb{R}$ and ϕ are both considered open and closed.
2. The union of any number of open sets is an open set; the intersection of any finite number of open sets is open.
3. The intersection of any number of closed sets is a closed set; the union of any finite number of closed sets is closed.
4. $\overline{(a,b)} = [a,b]$; if $A = \{1/n \mid n \in \mathbb{N}\}$, then $\overline{A} = A \cup \{0\}$.
5. Suppose f is continuous at $x = c$, and use Property 2 of the above Theorem. For all $\epsilon > 0$, $A = (f(c) - \epsilon, f(c) + \epsilon)$ is an open interval, so $f^{-1}(A)$ is an open set. Since $c \in f^{-1}(A)$, there is an open interval (a,b) such that $c \in (a,b)$. Pick $\delta = \min\{c-a, b-c\}$. Thus continuity of f at $x = c$ implies

$$\lim_{x \to c} f(x) = f(c).$$

Open Intervals and Open Sets in Σ

Using the distance function d, as defined in Section 9.2, we can define open sets in Σ.

1. An open interval of radius ϵ centered at $\mathbf{a} \in \Sigma$ will be the set

$$\{\mathbf{s} \in \Sigma \mid d[\mathbf{s}, \mathbf{a}] < \epsilon\}.$$

2. An open set in Σ is the union of any number of open intervals in Σ.

Note that if $\epsilon = 2^{-n}$, then $\mathbf{s}$ is in the open interval of radius ϵ centered at $\mathbf{a}$ if

$$s_i = a_i, \text{ for } 0 \le i \le n,$$

by the Proximity Theorem. That is, the first $n+1$ entries of $\mathbf{s}$ must be the same as the first $n+1$ entries of $\mathbf{a}$.

Definition of the Shift Map

Definition: The shift map $\sigma : \Sigma \longrightarrow \Sigma$ is defined by

$$\sigma(s_0 s_1 s_2 s_3 s_4 \ldots) = (s_1 s_2 s_3 s_4 \ldots).$$

That is, for $\mathbf{s} \in \Sigma$, $\sigma(\mathbf{s})$ is obtained from $\mathbf{s}$ by dropping its first entry. Thus:

1. $\sigma(010101\ldots) = (10101\ldots)$
2. $\sigma(011111\ldots) = (11111\ldots)$
3. $\sigma(001011\ldots) = (01011\ldots)$

Iterating the Shift Map

It is easy to iterate the shift map: we just keep dropping the first entry at each step. For example:

1. $\sigma^2(s_0 s_1 s_2 s_3 s_4 \ldots) = \sigma(s_1 s_2 s_3 s_4 \ldots) = (s_2 s_3 s_4 s_5 \ldots)$
2. $\sigma^3(s_0 s_1 s_2 s_3 s_4 \ldots) = \sigma(s_2 s_3 s_4 s_5 \ldots) = (s_3 s_4 s_5 s_6 \ldots)$
3. $\sigma^n(s_0 s_1 s_2 s_3 s_4 \ldots) = (s_n s_{n+1} s_{n+2} s_{n+3} \ldots)$

Notation: if

$$\mathbf{s} = (s_0 s_1 \ldots s_{n-1} s_0 s_1 \ldots s_{n-1} s_0 s_1 \ldots s_{n-1} \ldots)$$

is a repeating sequence, we shall write:

$$\mathbf{s} = (\overline{s_0 s_1 \ldots s_{n-1}}).$$

The Periodic Points of the Shift Map

If $\mathbf{s} = (\overline{s_0 s_1 \ldots s_{n-1}})$ is a repeating sequence then

$$\sigma^n(\mathbf{s}) = \mathbf{s}.$$

Conversely, any periodic point of period n for σ must be a repeating sequence. This is so much easier than it is for Q_c! We can actually write down all the periodic points for σ:

1. the only two fixed points for σ are $(11111\ldots)$ or $(00000\ldots)$
2. the only two points for σ of prime period 2 are $(\overline{01})$ or $(\overline{10})$, and they form a 2-cycle: $\sigma(\overline{01}) = (\overline{10})$, $\sigma(\overline{10}) = (\overline{01})$
3. there are only two 3-cycles for σ:

$$(\overline{001}) \to (\overline{010}) \to (\overline{100}) \to (\overline{001});$$

and

$$(\overline{110}) \to (\overline{101}) \to (\overline{011}) \to (\overline{110})$$

Continuity of the Shift Map

$\sigma : \Sigma \longrightarrow \Sigma$ is continuous in the sense that for any $\mathbf{s} \in \Sigma$, and for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$d[\mathbf{s}, \mathbf{t}] < \delta \Rightarrow d[\sigma(\mathbf{s}), \sigma(\mathbf{t})] < \epsilon.$$

Proof: pick n large enough so that $1/2^n < \epsilon$; pick $\delta = 1/2^{n+1}$. Then, by the Proximity Theorem,

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] < 1/2^{n+1} &\Rightarrow t_i = s_i, 0 \le i \le n+1 \\ &\Rightarrow \sigma(\mathbf{t}) = (s_1 s_2 s_3 \ldots s_{n+1} t_{n+2} t_{n+3} \ldots) \\ &\Rightarrow d[\sigma(\mathbf{s}), \sigma(\mathbf{t})] \le \frac{1}{2^n} < \epsilon, \end{aligned}$$

making use of the Proximity Theorem, again.

Connections, Connections ... and a Commuting Diagram

We have $Q_c : \Lambda \longrightarrow \Lambda$, $\sigma : \Sigma \longrightarrow \Sigma$ and $S : \Lambda \longrightarrow \Sigma$. What are the connections between these three functions?

Theorem: *If $x \in \Lambda$, then*

$$(S \circ Q_c)(x) = (\sigma \circ S)(x),$$

i.e. $S \circ Q_c = \sigma \circ S$ and the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{Q_c} & \Lambda \\ S \downarrow & & \downarrow S \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

Proof: let $x \in \Lambda$ and suppose $S(x) = (s_0 s_1 s_2 s_3 \dots)$. This means

$$Q_c^n(x) \in I_{s_n}, \text{ for } n \geq 0,$$

where I_{s_n} is either I_0 or I_1, depending on s_n. That is,

$$Q_c(x) \in I_{s_1}, Q_c^2(x) \in I_{s_2}, Q_c^3(x) \in I_{s_3}, \dots$$

and the itinerary of $Q_c(x)$ is $(s_1 s_2 s_3 \dots)$. Consequently

$$S(Q_c(x)) = (s_1 s_2 s_3 s_4 \dots) = \sigma(s_0 s_1 s_2 s_3 \dots) = \sigma(S(x)),$$

or equivalently

$$(S \circ Q_c)(x) = (\sigma \circ S)(x).$$

More Commuting Diagrams

$$\begin{array}{ccccc} \Lambda & \xrightarrow{Q_c} & \Lambda & \xrightarrow{Q_c} & \Lambda \\ S\big\downarrow & & \big\downarrow S & & \big\downarrow S \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

whence $S \circ Q_c^2 = \sigma^2 \circ S$.

$$\begin{array}{ccccccc} \Lambda & \xrightarrow{Q_c} & \Lambda & \xrightarrow{Q_c} & \Lambda & \xrightarrow{Q_c} & \Lambda \\ S\big\downarrow & & \big\downarrow S & & \big\downarrow S & & \big\downarrow S \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma & \xrightarrow[\sigma]{} & \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

whence $S \circ Q_c^3 = \sigma^3 \circ S$. In general $S \circ Q_c^n = \sigma^n \circ S$.

Orbits of x under Q_c and Orbits of $S(x)$ under σ

So S converts orbits of x under Q_c to orbits of $S(x)$ under σ. That is, S takes the orbit of x under Q_c, which is hard to calculate:

$$x, Q_c(x), Q_c^2(x), Q_c^3(x), \ldots, Q_c^n(x), \ldots$$

to

$$S(x), \sigma(S(x)), \sigma^2(S(x)), \sigma^3(S(x)), \ldots, \sigma^n(S(x)), \ldots,$$

which is easy to calculate, assuming you know what $S(x)$ is! Actually, we hope that by knowing things about orbits under σ we will be able to 'go the other way' and say something about the orbits under Q_c. But this requires some further properties of S.

Further Properties of $S : \Lambda \longrightarrow \Sigma$

The following properties of S can be proved:

1. S is one-to-one: $S(x) = S(y) \Rightarrow x = y$
2. S is onto: for each $\mathbf{s} \in \Sigma$ there is $x \in \Lambda$ such that $S(x) = \mathbf{s}$
3. S is continuous in the following sense: If $x \in \Lambda$, then for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in \Lambda, |y - x| < \delta \Rightarrow d[S(y), S(x)] < \epsilon.$$

4. $S^{-1} : \Sigma \longrightarrow \Lambda$ is also continuous.

$S : \Lambda \longrightarrow \Sigma$ is One-to-One

Suppose $x \neq y$ but $S(x) = S(y)$; then $Q_c^n(x)$ and $Q_c^n(y)$ are both in the same interval, I_0 or I_1, for each value of n. As we saw in Section 7.2, if

$$c \leq -\frac{5 + 2\sqrt{5}}{4},$$

then there is a number $\mu > 1$ such that the length of the interval $H = [Q_c^n(x), Q_c^n(y)]$ (or $[Q_c^n(y), Q_c^n(x)]$) is greater than

$$\mu^n |x - y|.$$

If n is large enough this length is greater than the length of I, contradicting the fact the interval H must be contained within I. Note: as in Section 7.2, this result is actually true for all $c < -2$.

$S : \Lambda \longrightarrow \Sigma$ is Onto; Use $F^{-1}(A \cap B) = F^{-1}(A) \cap F^{-1}(B)$

Let $\mathbf{s} \in \Sigma$; we shall construct $x \in \Lambda$ such that $S(x) = \mathbf{s}$. Let

$$\begin{aligned} I_{s_0 s_1 \dots s_n} &= \{x \in I \mid x \in I_{s_0}, Q_c(x) \in I_{s_1}, \dots, Q_c^n(x) \in I_{s_n}\} \\ &= I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap \cdots \cap Q_c^{-n}(I_{s_n}) \\ &= I_{s_0} \cap Q_c^{-1}\left(I_{s_1} \cap \cdots \cap Q_c^{-(n-1)}(I_{s_n})\right) \\ &= I_{s_0} \cap Q_c^{-1}(I_{s_1 s_2 \dots s_n}) \end{aligned}$$

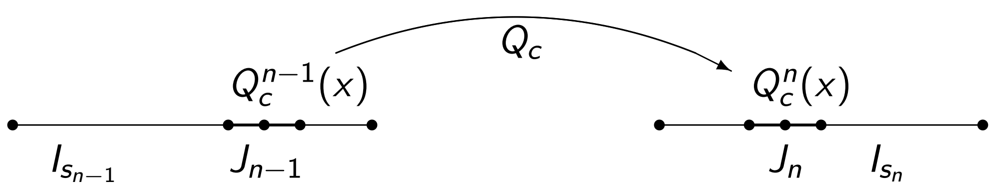
By induction, on the number of subscripts in $I_{s_0 s_1 \dots s_n}$, you can prove $I_{s_0 s_1 \dots s_n}$, is always a single closed interval. Moreover, these closed subintervals are nested because

$$I_{s_0 s_1 \dots s_n} = I_{s_0 s_1 \dots s_{n-1}} \cap Q_c^{-n}(I_{s_n}) \subset I_{s_0 s_1 \dots s_{n-1}}$$

Thus $\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n} \neq \phi$; and $x \in \bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n} \Rightarrow x \in \Lambda, S(x) = \mathbf{s}$.

$S : \Lambda \longrightarrow \Sigma$ is Continuous

Let $\epsilon > 0$ be given and pick n such that $1/2^n < \epsilon$.



Let J_n be a closed interval such that $J_n \subset I_{s_n}$, $Q_c^n(x) \in J_n$. Then $Q_c^{-1}(J_n)$ consists of two closed intervals (why?), one in I_0 and one in I_1; let J_{n-1} be the closed interval in $I_{s_{n-1}}$ that contains $Q_c^{n-1}(x)$. Proceed in this fashion for $0 \leq i \leq n$ to obtain closed intervals $J_i \subset I_{s_i}$, such that $Q_c^i(x) \in J_i$. Then

$$x, y \in J_0 \Rightarrow Q(x), Q(y) \in J_1 \Rightarrow \cdots \Rightarrow Q_n^n(x), Q_c^n(y) \in J_n.$$

Consequently, $x, y \in \Lambda \cap J_0 \Rightarrow d[S(x), S(y)] \leq 1/2^n < \epsilon$.

$S^{-1} : \Sigma \longrightarrow \Lambda$ Exists

Since $S : \Lambda \longrightarrow \Sigma$ is one-to-one and onto, $S^{-1} : \Sigma \longrightarrow \Lambda$ exists.

$$\begin{array}{ccc} \Lambda & \xrightarrow{Q_c} & \Lambda \\ {\scriptstyle S^{-1}}\uparrow & & \uparrow{\scriptstyle S^{-1}} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

Hence $Q_c \circ S^{-1} = S^{-1} \circ \sigma$. In general,

$$Q_c^n \circ S^{-1} = S^{-1} \circ \sigma^n.$$

Thus there is a one-to-one correspondence between the orbits of $x \in \Lambda$ under Q_c and the orbits of $\mathbf{s} \in \Sigma$ under σ.

S^{-1} Is Also Continuous

S^{-1} is continuous, in the sense that for any $\mathbf{s} \in \Sigma$, and all $\epsilon > 0$, there is a $\delta > 0$, such that $d[\mathbf{s}, \mathbf{t}] < \delta \Rightarrow |S^{-1}(\mathbf{s}) - S^{-1}(\mathbf{t})| < \epsilon$.
Proof: let $\mathbf{s} = S(x), \mathbf{t} = S(y)$. Then $d[\mathbf{s}, \mathbf{t}] < 1/2^n$ means $Q_c^i(x)$ and $Q_c^i(y)$ are always in the same interval I_0 or I_1 for $0 \le i \le n$. As we saw in Section 7.2, for $c < -2.368$, there is $\mu > 1$ such that

$$|Q_c^n(x) - Q_c^n(y)| \ge \mu^n |x - y| \Leftrightarrow |x - y| \le \frac{|Q_c^n(x) - Q_c^n(y)|}{\mu^n}$$

But $|Q_c^n(x) - Q_c^n(y)| < L$, where L is the length of I_1. (Why?) Pick n large enough so that $L/\mu^n < \epsilon$. Then

$$d[S(x), S(y)] < 1/2^n \Rightarrow |x - y| < \epsilon \Leftrightarrow |S^{-1}(S(x)) - S^{-1}(S(y))| < \epsilon.$$

What Is a Homeomorphism?

Suppose $h : X \longrightarrow Y$ and d_1 and d_2 are distance functions defined on X and Y, respectively. Then h is called a homeomorphism if

1. h is one-to-one: $h(x) = h(y) \Rightarrow x = y$
2. h is onto: if $y \in Y$ there is an $x \in X$ such that $h(x) = y$
3. h is continuous on X: for all $x \in X$ and all $\epsilon > 0$, there is a $\delta > 0$ such that $d_1[x, y] < \delta \Rightarrow d_2[h(x), h(y)] < \epsilon$
4. $h^{-1} : Y \longrightarrow X$ is also continuous.

Equivalently: $h : X \longrightarrow Y$ is a homeomorphism if it is a bijection that maps open sets to open sets, and closed sets to closed sets.
Consequently: x_1 and x_2 are close together in X if and only if $h(x_1)$ and $h(x_2)$ are close together in Y.
Examples: $S : \Lambda \longrightarrow \Sigma$ is a homeomorphism. More mundanely, $h(x) = x^3$, $h : \mathbb{R} \longrightarrow \mathbb{R}$, is a homeomorphism.

Conjugacy

Definition: Let $F : X \longrightarrow X$ and $G : Y \longrightarrow Y$ be two functions. F and G are called conjugate if there is a homeomorphism $h : X \longrightarrow Y$ such that $h \circ F = G \circ h$; h is called a conjugacy.

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow[G]{} & Y \end{array}$$

$$h \circ F = G \circ h.$$

$$\begin{aligned} F^n &= h^{-1} \circ G^n \circ h \\ G^n &= h \circ F^n \circ h^{-1} \end{aligned}$$

The Conjugacy Theorem

We have proved

> **Theorem:** *The shift map σ on Σ is conjugate to Q_c on Λ; the conjugacy is S.*

$$\begin{array}{ccc} \Lambda & \xrightarrow{Q_c} & \Lambda \\ S\downarrow & & \downarrow S \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

$$S \circ Q_c = \sigma \circ S$$

$$\begin{aligned} Q_c^n &= S^{-1} \circ \sigma^n \circ S \\ \sigma^n &= S \circ Q_c^n \circ S^{-1} \end{aligned}$$

This means that dynamics of σ on Σ and Q_c on Λ are essentially the same. For example:

1. S converts orbits of x under Q_c to orbits of $S(x)$ under σ
2. S^{-1} converts orbits of $\mathbf{s}$ under σ to orbits of $S^{-1}(\mathbf{s})$ under Q_c
3. If $\mathbf{s}$ is a periodic point for σ then $S^{-1}(\mathbf{s})$ is a periodic point for Q_c, with the same period.
4. If $\mathbf{s}$ is an eventually periodic point for σ then $S^{-1}(\mathbf{s})$ is an eventually periodic point for Q_c

Eg. $\sigma(\mathbf{s}) = \mathbf{s} \Rightarrow S^{-1}(\sigma(\mathbf{s})) = S^{-1}(\mathbf{s}) \Rightarrow Q_c(S^{-1}(\mathbf{s})) = S^{-1}(\mathbf{s})$.

How Can We Define Chaos?

If something is truly chaotic then it should defy all definition or description. A chaotic dynamical system appears chaotic in the colloquial sense, but actually satisfies some definite criteria in the mathematical sense. Those three criteria are

1. density
2. transitivity
3. sensitivity

First we shall define these three notions, and then we shall give some examples of dynamical systems that satisfy these three criteria.

What Is a Dense Set?

Definition: Suppose Y is a subset of X, which has distance function d. Y is said to be dense in X if for every open interval $A \subset X$,

$$Y \cap A \neq \phi.$$

Equivalently:

1. if $x \in X$ then there is a sequence $y_n \in Y$ such that
$$\lim_{n\to\infty} y_n = x.$$
2. $\overline{Y} = X$; i.e. the closure of Y is X.

What is open, or closed, and the calculation of limits, all depend on the distance function d. Intuitively, Y is a dense subset of X if for any point $x \in X$ there is a point $y \in Y$ arbitrarily close to x.

Examples of Dense Sets in $\mathbb{R}$

For $\mathbb{R}$ the usual distance function is absolute value: the distance between $a, b \in \mathbb{R}$ is $|a - b|$; and a typical open interval in $\mathbb{R}$ is (a, b). Examples:

1. The rational numbers $\mathbb{Q}$ are dense in the real numbers $\mathbb{R}$. **Proof:** suppose $x = a_j \dots a_1 a_0 \cdot b_1 b_2 \dots b_n \dots$ is an irrational number. Then the sequence of $y_n = a_j \dots a_1 a_0 \cdot b_1 b_2 \dots b_n$ is a sequence of rational numbers with limit x. (Think of

$$\sqrt{2} = 1.4142135623730950488\dots;$$

take $y_1 = 1.4, y_2 = 1.41, y_3 = 1.414$, and so on.)
2. (a, b) is dense in $[a, b]$
3. The integers $\mathbb{Z}$ are not dense in $\mathbb{R}$ or $\mathbb{Q}$; no sequence of integers can have limit $1/2$, for instance.

Examples of Dense Sets in Σ

Example 1: The set of periodic points in Σ is dense in Σ: Let $\mathbf{s} = (s_0 s_1 \dots s_n s_{n+1} \dots)$ be an arbitrary sequence in Σ, let $\epsilon > 0$. Pick n such that $1/2^n < \epsilon$; let $\mathbf{t_n} = (\overline{s_0 s_1 \dots s_n})$. Then $\mathbf{t_n}$ is a periodic point in Σ and, by the Proximity Theorem,

$$d[\mathbf{s}, \mathbf{t_n}] \le 1/2^n < \epsilon.$$

Example 2: The orbit of

$$\hat{\mathbf{s}} = (\underbrace{0\,1}_{\text{all 1 blocks}}\ \underbrace{00\,01\,10\,11}_{\text{all 2 blocks}}\ \underbrace{000\,010\,101\,\dots}_{\text{all 3 blocks}}\dots)$$

under σ is dense in Σ: Given $\mathbf{s} \in \Sigma$ pick k such that the $n+1$ block at the beginning of $\sigma^k(\hat{\mathbf{s}})$ is the same as $(s_0 s_1 \dots s_n)$. Now apply the Proximity Theorem.

What Is Transitivity?

Definition: A dynamical system $F : X \longrightarrow X$ is transitive if for any points $x, y \in X$ and any $\epsilon > 0$, there is a third point z, within ϵ of x, whose orbit comes within ϵ of y. That is, for any $x, y \in X$ there is a $z \in X$ and an $n > 0$ such that

$$d[z, x] < \epsilon \text{ and } d[F^n(z), y] < \epsilon.$$

In other words, a transitive dynamical system has the property that there is always an orbit that gets arbitrarily close to any two given points.

Example

$\sigma : \Sigma \longrightarrow \Sigma$ is transitive, because as we have seen, the orbit of

$$\hat{\mathbf{s}} = (\underbrace{0\,1}_{\text{all 1 blocks}}\ \underbrace{00\,01\,10\,11}_{\text{all 2 blocks}}\ \underbrace{000\,010\,101\,\ldots}_{\text{all 3 blocks}}\ \ldots)$$

under σ is dense in Σ. So, if $\mathbf{s}, \mathbf{t} \in \Sigma$, and $\epsilon > 0$,

1. then there is a k such that $d[\mathbf{s}, \sigma^k(\hat{\mathbf{s}})] < \epsilon$
2. and some n steps later, $d[\mathbf{t}, \sigma^{k+n}(\hat{\mathbf{s}})] < \epsilon$

In fact, any dynamical system that has a dense orbit must be transitive.

What Is Sensitivity to Initial Conditions?

Definition: A dynamical system $F : X \longrightarrow X$ depends sensitively on initial conditions if there is a $\beta > 0$ such that for every $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ within ϵ of x and a k such that the distance between $F^k(x)$ and $F^k(y)$ is at least β. That is, for every $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ and a k such that

$$d[y, x] < \epsilon \text{ but } d[F^k(y), F^k(x)] \geq \beta.$$

We can find y as close to x as we like but eventually the orbit of x and y under F will be separated by more than β. Putting it yet another way: there is no guarantee that choosing x and y close together will ensure the orbits of x and y under F stay close together. In terms of numerical calculations: a small round-off error can result in a totally different orbit being calculated.

Example of a Sensitive Dynamical System

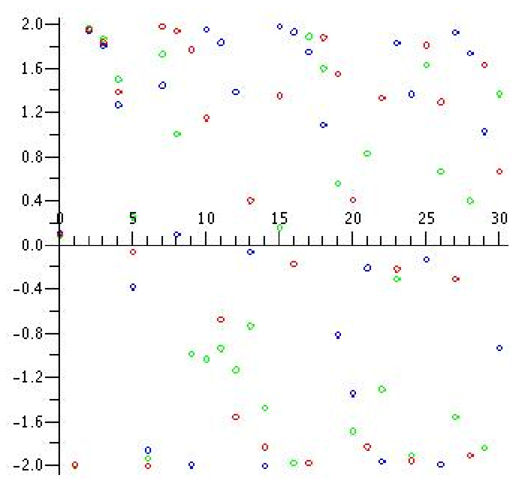
We shall see shortly that

$$Q_{-2} : [-2, 2] \longrightarrow [-2, 2]$$

is a sensitive dynamical system, with of course

$$Q_{-2}(x) = x^2 - 2.$$

The diagram to the right illustrates the first 30 iterations for the three orbits of $x = 0.09, 0.1, 0.11$

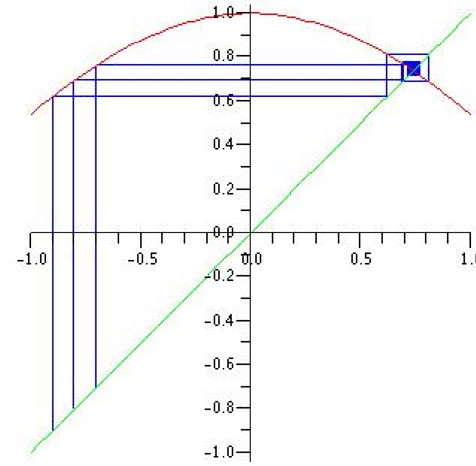


Example of an Insensitive Dynamical System

$C : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$C(x) = \cos x$$

possesses no sensitivity to initial conditions whatsoever. As we have seen, for any choice of x the orbit of x under C converges to the unique fixed point of C, approximately 0.73908. Orbits that start together stay together. The diagram to the right illustrates three orbits for $x = -.9, -.8, -.7$



Definition of A Chaotic Dynamical System

Definition: A dynamical system $F : X \longrightarrow X$ is chaotic if

1. The set of all periodic points for F is dense in X.
2. F is transitive.
3. F depends sensitively on initial conditions.

Example: The shift map $\sigma : \Sigma \longrightarrow \Sigma$ is chaotic. We have already proved properties 1 and 2; we only need to show that σ depends sensitively on initial conditions.

$\sigma : \Sigma \longrightarrow \Sigma$ Depends Sensitively On Initial Conditions

Let $\beta = 1$; for any $\mathbf{s} \in \Sigma$, and any $\epsilon > 0$, pick n large enough so that $1/2^n < \epsilon$. Suppose $\mathbf{t} \in \Sigma$ and $d[\mathbf{t}, \mathbf{s}] < 1/2^n$. If $\mathbf{t} \neq \mathbf{s}$, then there is a $k > n$ such that $t_k \neq s_k$. Thus $|t_k - s_k| = 1$ and consequently

$$d[\sigma^k(\mathbf{t}), \sigma^k(\mathbf{s})] = \sum_{j=0}^{\infty} \frac{|t_{k+j} - s_{k+j}|}{2^j} \geq |t_k - s_k| = 1.$$

This actually proves more than sensitivity to initial conditions of the shift map; it proves that all other points $\mathbf{t} \neq \mathbf{s}$ have orbits that eventually separate by at least 1 from the orbit of $\mathbf{s}$ under σ.

The Density Proposition

Before we can prove that Q_c is chaotic on Λ we need one more preliminary result.

> **Theorem:** *Suppose $F : X \longrightarrow Y$ is a continuos map that is onto and suppose that D is a dense subset of X. Then $F(D)$ is a dense subset of Y.*

Proof: Let B be an open set in Y. Since F is onto and continuous, $A = F^{-1}(B)$ is a non-empty open set in X; so $A \cap D \neq \phi$. Pick $x \in A \cap D$. Then $F(x) \in F(A) \cap F(D) = B \cap F(D)$. Hence

$$B \cap F(D) \neq \phi.$$

$Q_c : \Lambda \longrightarrow \Lambda$ Is A Chaotic Dynamical System, If $c < -2$

$$\begin{array}{ccc} \Lambda & \xrightarrow{Q_c} & \Lambda \\ S^{-1}\big\uparrow & & \big\uparrow S^{-1} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

$S^{-1} : \Sigma \longrightarrow \Lambda$ is also a homeomorphism, and $S^{-1}(\mathbf{s})$ is a periodic point in Λ if and only if $\mathbf{s}$ is a periodic point in Σ. Then, by the Density Proposition, S^{-1} maps the dense set of periodic points for σ in Σ to a dense set of periodic points for Q_c in Λ.

Secondly, since the orbit of $\hat{\mathbf{s}}$ under σ is dense in Σ, the Density Proposition ensures that the orbit of $S^{-1}(\hat{\mathbf{s}})$ under Q_c is also dense in Λ. This means Q_c is also transitive.

It only remains to show that Q_c is sensitive to initial conditions.

$Q_c : \Lambda \longrightarrow \Lambda$ Is Sensitive to Initial Conditions

$$\begin{array}{ccc} \Lambda & \xrightarrow{Q_c} & \Lambda \\ S\big\downarrow & & \big\downarrow S \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

Recall that I_0 and I_1 are the closed disjoint subintervals of $I = [-p_+, p_+]$, produced by discarding all the points in the open interval A_1. Let β be the length of the interval A_1. Now let $x, y \in \Lambda, x \neq y$.

Since S is bijective $S(x) \neq S(y)$, and there is a k such that the k-th entries of $S(x)$ and $S(y)$ differ. This means that both $F^k(x)$ and $F^k(y)$ are not in the same interval, I_0 or I_1. Consequently,

$$|F^k(x) - F^k(y)| \geq \beta.$$

Therefore the orbit of y under Q_c for any $y \neq x$ eventually separates from the orbit of x under Q_c by at least β.

Another Chaotic Dynamical System

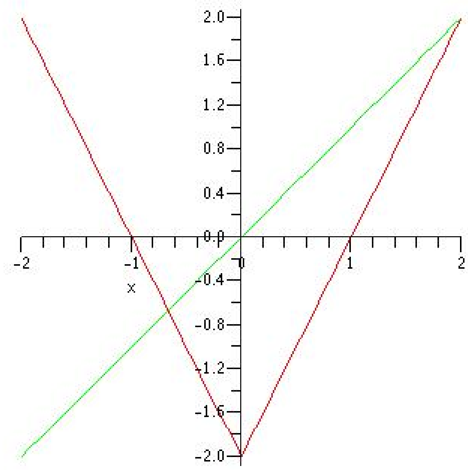
Let

$$V(x) = 2|x| - 2.$$

Check that

$$V : [-2, 2] \longrightarrow [-2, 2];$$

or look at the graph of V to the right.



We claim that $V : [-2, 2] \longrightarrow [-2, 2]$ is a chaotic dynamical system. To see why we need only consider graphs of V^n.

Chapter 9: Symbolic Dynamics
Chapter 10: Chaos
10.1 The Three Properties of a Chaotic System
10.2 Other Chaotic Systems

Higer Iterations of V

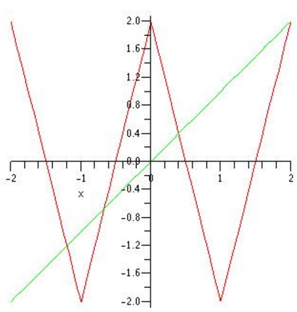


Figure: V^2

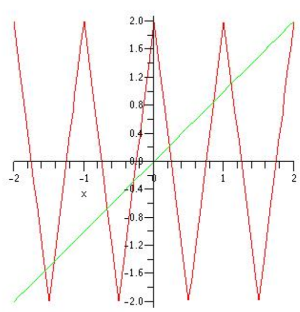


Figure: V^3

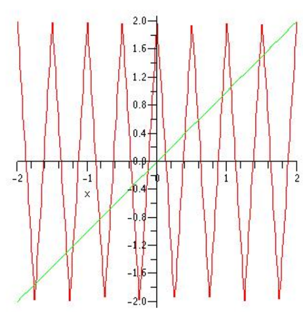


Figure: V^4

In general, the graph of V^n maps $[-2, 2]$ to itself, consists of 2^n line segments with slope $\pm 2^n$, each of which maps an interval of length $1/2^{n-2}$ onto the interval $[-2, 2]$.

Chapter 9 and 10 Lecture Notes
MAT335H1F Lec0101 Burbulla

Periodic Points of V are Dense in $[-2,2]$

Let J be an arbitrary open interval in $[-2,2]$. Pick a closed interval J_n of length $1/2^{n-2}$ such that $J_n \subset J$ and J_n is one of the subintervals in $[-2,2]$ on which

$$V^n : J_n \longrightarrow [-2,2].$$

Since every segment of V^n intersects the line $y = x$, there is an $x \in J_n$ such that

$$V^n(x) = x.$$

Since $J_n \subset J$, J contains a periodic point of V.

$V : [-2,2] \longrightarrow [-2,2]$ Is Transitive

Pick any two points $x, y \in [-2,2]$, and let $\epsilon > 0$ be given. Pick n large enough so that $1/2^{n-2} < \epsilon$. As in the previous slide, pick a closed interval J_n of length $1/2^{n-2}$ such that $x \in J_n$ and

$$V^n : J_n \longrightarrow [-2,2].$$

Since $y \in [-2,2]$, there is a $z \in J_n$ such that $V^n(z) = y$. Then

$$|x - z| < 1/2^{n-2} < \epsilon \text{ and } |y - V^n(z)| = 0 < \epsilon.$$

So not only does the orbit of z under V get close to y, it actually hits y. In any event, V is transitive on $[-2,2]$.

V Depends Sensitively on Initial Conditions

Take $\beta = 2$, let $\epsilon > 0$ and suppose $x \in [-2, 2]$. Pick n large enough so that $1/2^{n-2} < \epsilon$. As in the previous two slides, pick a closed interval J_n of length $1/2^{n-2}$ such that $x \in J_n$ and

$$V^n : J_n \longrightarrow [-2, 2].$$

Pick $y \in J_n$ such that

$$|y - x| \geq \frac{1}{2} \text{ length } (J_n) = \frac{1}{2^{n-1}},$$

which is possible because x lies in one half of J_n. Now apply MVT to V^n on the interval $[x, y]$ (or $[y, x]$): there is $c \in [x, y]$ such that

$$|V^n(y) - V^n(x)| = |(V^n)'(c)||y - x| \geq \frac{2^n}{2^{n-1}} = 2.$$

Is $Q_{-2} : [-2, 2] \longrightarrow [-2, 2]$ A Chaotic Dynamical System?

Let $C(x) = -2cos(\pi x/2)$. Then $Q_{-2} \circ C = C \circ V$:

$$\begin{array}{ccc} [-2,2] & \xrightarrow{V} & [-2,2] \\ C\downarrow & & \downarrow C \\ [-2,2] & \xrightarrow[Q_{-2}]{} & [-2,2] \end{array}$$

$$\begin{aligned} C(V(x)) &= C(2|x| - 2) \\ &= -2\cos(\pi|x| - \pi) \\ &= 2\cos(\pi x), \end{aligned}$$

making use of some trig, and

$$Q_{-2}(C(x)) = Q_{-2}(-2\cos(\pi x/2)) = 4\cos^2(\pi x/2) - 2 = 2\cos(\pi x).$$

It seems as if C is a conjugacy, whence Q_{-2} is chaotic, since V is. Actually C is not one-to-one, but it is continuous and onto, so the Density Proposition still applies to C. Although C is two-to-one, it still takes periodic points of V to periodic points of Q_{-2}. So C can be used to prove that Q_{-2} is chaotic. C is called a semiconjugacy.

Semiconjugacy

Suppose $F : X \longrightarrow X$ and $G : Y \longrightarrow Y$ are two dynamical systems. A mapping $h : X \longrightarrow Y$ is called a semiconjugacy if h is continuous, onto, at most n-to-one, and satisfies $h \circ F = G \circ h$.

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow[G]{} & Y \end{array}$$

Then $h(F^k(x)) = G^k(h(x))$. Thus a semiconjugacy takes orbits of x under F to orbits of $h(x)$ under G; and takes cycles of F to cycles of G, although its prime period may become less.

Since h is continuous and onto, the Density Proposition ensures the periodic points of G are dense in Y, if the periodic points of F are dense in X. Similarly, if F has a dense orbit in X, then G has a dense orbit in Y, although it may have many less distinct points. Semiconjugacies do usefully relate one system to another.

Another Doubling Function, Reference Page 125

Devaney calls it D, but I'm going to call it D_2, because it is just the square function . . . applied to a complex variable z. That is:

1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$
2. Define $D_2 : S^1 \longrightarrow S^1$ by
$$D_2(z) = z^2.$$
3. The orbit of $z = i$ under D_2 is eventually fixed: $i, -1, 1, 1, \ldots$
4. The orbit of $z = (1 + \sqrt{3}\, i)/2$ under D_2 is eventually a 2-cylce, as shown on the figure.

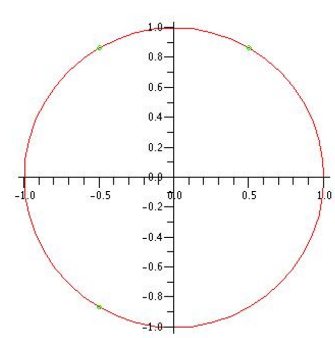


Figure: Unit Circle in $\mathbb{C}$

Why Does Devaney Call D_2 a Doubling Function?

If $z = \cos\theta + i\sin\theta \in S^1$, then $z^2 = \cos(2\theta) + i\sin(2\theta)$. So the argument θ of z is doubling. Now consider $B : S^1 \longrightarrow [-2, 2]$ defined by $B(z) = 2\text{Re}\,(z)$. Check that $B \circ D_2 = Q_{-2} \circ B$:

$$\begin{aligned} Q_{-2}(B(z)) &= Q_{-2}(2\cos\theta) \\ &= 4\cos^2\theta - 2 \\ &= 2\cos(2\theta) \\ &= B(z^2) \end{aligned}$$

$$\begin{array}{ccc} S^1 & \xrightarrow{D_2} & S_1 \\ B\downarrow & & \downarrow B \\ [-2,2] & \xrightarrow[Q_{-2}]{} & [-2,2] \end{array}$$

Since B is two-to-one, B is a semiconjugacy; we can use it to show D_2 is chaotic on S^1 since Q_{-2} is chaotic on $[-2, 2]$. But it's tricky since we have to use B^{-1}, not a function, to get information about D_2 from Q_{-2}. (See Chapter 16 for a more direct approach.)

Using Some Properties of Complex Numbers

Note: z is a periodic point for D_2 iff $\overline{z}$ is; and $B(z) = B(\overline{z})$.
Density: suppose the periodic points of D_2 are not dense in S^1. Then there is an open interval $U \subset S^1$ that contains no periodic point for D_2. But $B(U)$ is an open interval in $[-2, 2]$ and so must contain a periodic point for Q_{-2}, say x. Let $B^{-1}(x) = \{z, \overline{z}\}$. Then

$$B(D_2^n(z)) = Q_{-2}^n(B(z)) = Q_{-2}^n(x) = x \Rightarrow D_2^n(z) = z \text{ or } \overline{z}.$$

If $D_2^n(z) = z$ then z is a periodic point; if $D_2^n(z) = \overline{z}$, then $D_2^{2n}(z) = z$; either way z and $\overline{z}$ are periodic points for D_2. By continuity, $B^{-1}(B(U))$ is an open set, $U \cup \overline{U}$, and one of z or $\overline{z}$ is in U, contradicting our assumption that U contains no periodic points. Proving **Transitivity** and **Sensitivity** is left to the reader.