MAT335H1F Lec0101 Burbulla

Chapter 11, 12 and 13 Lecture Notes

Fall 2012

Chapter 11: Sarkovskii's Theorem

Chapter 12: The Role of the Critical Orbit

Chapter 13: Newton's Method

The Period 3 Theorem

Theorem: *Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and has a periodic point of prime period 3. Then F has periodic points of all other periods.*

This theorem was published in *American Mathematical Monthly* **82** (1975), pp 985-992 by Li, T.-Y. and Yorke, J. under the title "Period Three Implies Chaos." It was the first appearance in the scientific literature of the word "chaos." It is actually a special case of Sarkovskii's Theorem which was first proved in 1964, but was unknown at the time in the West. We will look at Sarkovskii's Theorem in Section 11.2, but we will not prove it. Instead, we will only prove the easier Period 3 Theorem.
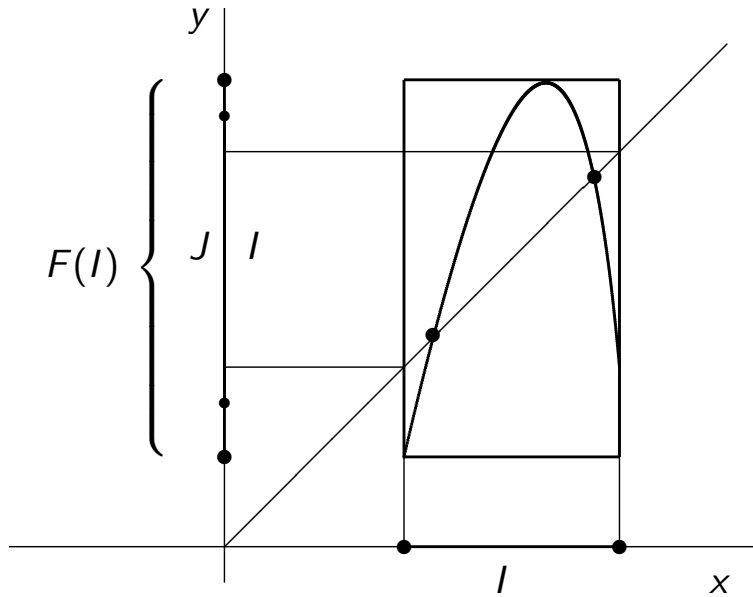
Two Observations

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

1. Suppose I and J are closed intervals in $\mathbb{R}$ and $I \subset J$. If $J \subset F(I)$, then F has a fixed point in I.

2. Suppose I and J are two closed intervals in $\mathbb{R}$ and $J \subset F(I)$. Then there is a closed subinterval $I' \subset I$ such that $F(I') = J$.
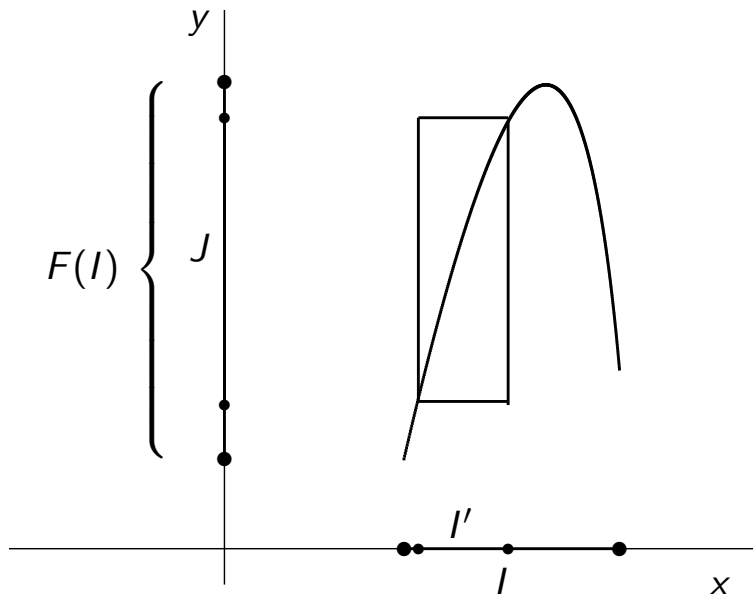
Proof of Observation One, By Diagram

$$I \subset J \subset F(I) \Rightarrow F \text{ has a fixed point in } I$$

Proof of Observation Two, By Diagram

$$J \subset F(I) \Rightarrow F(I') = J, \text{ for some } I' \subset I$$

Proof of the Period 3 Theorem

Suppose $a \to b \to c \to a$ under F and that a is the least of the three values. Then $a < b < c$ or $a < c < b$. Assume $a < b < c$; the other case is handled similarly. Let $I_0 = [a, b]$, $I_1 = [b, c]$. By continuity:

$$F(a) = b, F(b) = c \implies I_1 \subset F(I_0), \text{ and}$$
$$F(b) = c, F(c) = a \implies F(I_1) \supset [a, c] = I_0 \cup I_1.$$

Since $I_1 \subset F(I_1)$, F has a fixed point in I_1, by Obs 1. By Obs 2, there is a closed interval $A_1 \subset I_1$ such that $F(A_1) = I_1$. Now $A_1 \subset F(A_1)$ implies there is a closed interval $A_2 \subset A_1$ such that $F(A_2) = A_1$, again by Obs 2. Continue in this way, $n - 2$ times, to produce a collection of closed nested intervals, A_i, $1 \leq i \leq n - 2$, such that $A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1$, $F(A_i) = A_{i-1}$, and $F(A_1) = I_1$. In particular, $A_{n-2} \subset I_1$ and $F^{n-2}(A_{n-2}) = I_1$.
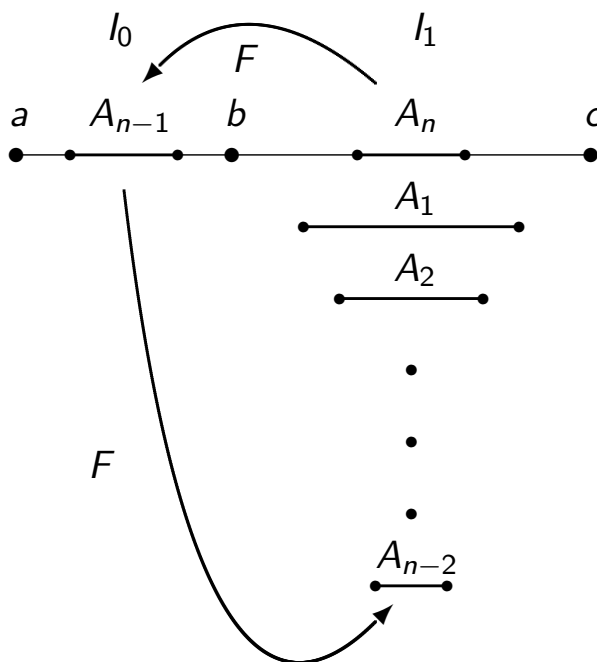
Since $A_{n-2} \subset I_1 \subset F(I_0)$ there is a closed subinterval $A_{n-1} \subset I_0$ such that $F(A_{n-1}) = A_{n-2}$. And since $A_{n-1} \subset I_0 \subset F(I_1)$, there is a closed interval $A_n \subset I_1$ such that $F(A_n) = A_{n-1}$. Altogether we have:

$$F^n(A_n) = I_1.$$

Since $A_n \subset I_1$ we can apply Obs 1 to conclude that there is a point $x_0 \in A_n$ such that $F^n(x_0) = x_0$.

We claim x_0 is of prime period n: $x_1 \in I_0$ but all other $x_i \in I_1$, for $i > 1$. Thus the period of x_0 is equal to n.

The Sarkovskii Ordering

The following ordering of the natural numbers

$$3, 5, 7, 9, 11, 13, 15, \ldots$$
$$6, 10, 14, 18, 22, 26, 30, \ldots$$
$$12, 20, 28, 36, 44, 52, 60, \ldots$$
$$24, 40, 56, 72, 88, 104, 120, \ldots$$
$$\ldots \ldots$$
$$\ldots \ldots$$
$$\ldots, 2^n, \ldots, 8, 4, 2, 1$$

is called the Sarkovskii ordering.

If q stands for an odd number, then the pattern is

q	ascending order
$2\, q$	ascending order
$2^2\, q$	ascending order
$2^3\, q$	ascending order
$\ldots$	$\ldots$
$\ldots$	$\ldots$
2^n	descending order

for $n \geq 0$.

Sarkovskii's Theorem

Theorem: *Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Suppose that F has a periodic point of prime period n and that n precedes k in the Sarkovskii ordering. Then F also has a periodic point of prime period k.*

Comments:

1. The first number in the Sarkovskii ordering is 3, so the Period Three Theorem is a corollary of Sarkovskii's Theorem.

2. The only assumption is that F is continuous.

3. Sarkovskii's Theorem also applies to a continuous function $F : [a, b] \longrightarrow [a, b]$; just extend F to all of $\mathbb{R}$ by $F(x) = F(a)$ if $x < a$, and $F(x) = F(b)$, if $x > b$.

## Some Applications of Sarkovskii's Theorem

1. If F has a periodic point of prime period 6 then it must have periodic points of all other even prime periods, but it may not have any periodic points with odd prime period.

2. If F has a periodic point of prime period 56 then it must have a periodic point of prime period 48, since $56 = 2^3 \cdot 7$ is before $48 = 2^4 \cdot 3$ in the Sarkovskii ordering.

3. If F has only finitely many periodic points then their prime periods must all be a power of 2.

4. If F is increasing it has no periodic points of prime order greater than 1, because increasing functions have no points of prime period 2, which is the second to last entry in the Sarkovskii ordering.

The Converse of Sarkovskii's Theorem

Theorem: *For any number n there is a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that F has a periodic point of prime period n but has no periodic points of any prime period k preceding n in the Sarkovskii ordering.*

Or putting it another way, for any n you can find a function F that has an n-cycle but has no k-cycles for any number k that precedes n in the Sarkovskii ordering.

Example
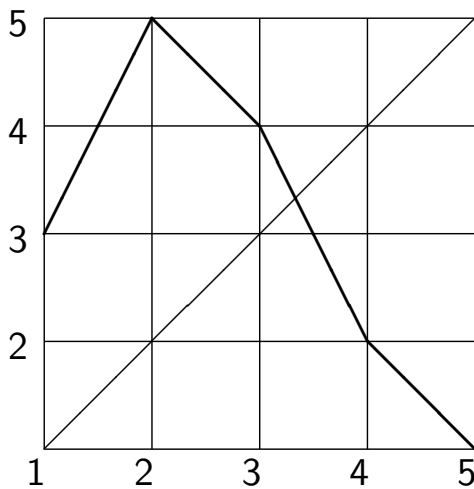
The graph of

$$F : [1, 5] \longrightarrow [1, 5]$$

is given to the left. We claim F has a 5-cycle but no 3-cycle. The 5-cycle is

$$1 \to 3 \to 4 \to 2 \to 5 \to 1,$$

as you may check.

We show F has no 3-cycle on the next slide.

From the graph

$$F([1, 2]) = [3, 5], \ F([3, 5]) = [1, 4], \ \text{and } F([1, 4]) = [2, 5],$$

so $F^3([1, 2]) = [2, 5]$, and the only fixed point of F^3 in $[1, 2]$ could be 2. But 2 is a point of period 5, so F^3 has no fixed point in $[1, 2]$. Similarly you can show that F^3 has no fixed points in the intervals $[2, 3]$ or $[4, 5]$. We need to alter our argument for F on $[3, 4]$ since F has a fixed point in this interval. Again from the graph we have

$$F([3, 4]) = [2, 4], \ F([2, 4]) = [2, 5], \ \text{and } F([2, 5]) = [1, 5],$$

so $F^3([3, 4]) = [1, 5]$. Additionally: F is decreasing on each of the intervals $[3, 4], [2, 4]$ and $[2, 5]$, so F^3 is also decreasing on $[3, 4]$. Thus there is exactly one solution to the equation $F^3(x) = x$ in $[3, 4]$: the fixed point of F, not a point of prime period 3.

What Is The Schwarzian Derivative?

Definition: The Schwarzian derivative of a function F is

$$S(F)(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2.$$

Examples:

1. $S(Q_c)(x) = 0 - \dfrac{3}{2}\left(\dfrac{2}{2x}\right)^2 = -\dfrac{3}{2x^2} < 0$

2. $E(x) = e^x \Rightarrow S(E)(x) = \dfrac{e^x}{e^x} - \dfrac{3}{2}\left(\dfrac{e^x}{e^x}\right)^2 = -\dfrac{1}{2} < 0$

3. $S(\sin)(x) = \dfrac{-\cos x}{\cos x} - \dfrac{3}{2}\left(\dfrac{-\sin x}{\cos x}\right)^2 = -1 - \dfrac{3}{2}\tan^2 x < 0$

The Schwarzian Derivative of a Polynomial

Proposition: Suppose $P(x)$ is a polynomial and all roots of $P'(x)$ are real and distinct. Then $S(P)(x) < 0$ for all x.

Proof: Let $P'(x) = k(x - a_1)(x - a_2) \cdots \cdots (x - a_n)$. Differentiate logarithmically:

$$\frac{P''(x)}{P'(x)} = \sum_{i=1}^{n} \frac{1}{x - a_i}.$$

Differentiate both side with respect to x:

$$\frac{P'''(x)}{P'(x)} - \left(\frac{P''(x)}{P'(x)}\right)^2 = -\sum_{i=1}^{n} \frac{1}{(x - a_i)^2}.$$

Then $S(P)(x) =$

$$\frac{P'''(x)}{P'(x)} - \left(\frac{P''(x)}{P'(x)}\right)^2 - \frac{1}{2}\left(\frac{P''(x)}{P'(x)}\right)^2 = -\sum_{i=1}^{n} \frac{1}{(x - a_i)^2} - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{x - a_i}\right)^2 < 0.$$

Chain Rule for Schwarzian Derivatives

Suppose F and G are functions. Then

$$S(F \circ G)(x) = S(F)(G(x)) \cdot (G'(x))^2 + S(G)(x)$$

Outline of Proof: By the usual chain rule, we find:

$$(F \circ G)'(x) = F'(G(x)) \cdot G'(x),$$

$$(F \circ G)''(x) = F''(G(x)) \cdot (G'(x))^2 + F'(G(x)) \cdot G''(x),$$

and $(F \circ G)'''(x) =$

$$F'''(G(x)) \cdot (G'(x))^3 + 3F''(G(x)) \cdot G''(x) \cdot G'(x) + F'(G(x)) \cdot G'''(x).$$

Now substitute into the formula for $S(F \circ G)(x)$ ... and simplify!

Corollary: If $S(F) < 0$ and $S(G) < 0$, then $S(F \circ G) < 0$. In particular, if $S(F) < 0$, then $S(F^n) < 0$, for all $n > 1$.

Proof: We have $S(F)(G(x)) < 0$ and $S(G)(x) < 0$ for all x. Thus

$$S(F \circ G)(x) = S(F)(G(x)) \cdot (G'(x))^2 + S(G)(x) < 0 + 0 = 0.$$

Schwarzian Min-Max Principle

Theorem: *Suppose $S(F) < 0$. Then F' cannot have a positive local minimum value or a negative local maximum value.*

Proof: Suppose c is a critical point of F', so that $F''(c) = 0$. If $S(F)$ is defined at $x = c$, then $F'(c) \neq 0$ and
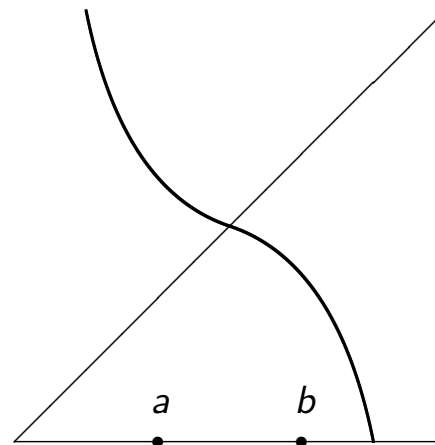
$$S(F)(c) = \frac{F'''(c)}{F'(c)}.$$

If F' has a positive local minimum value at $x = c$ then *its* second derivative, F''', must be positive at $x = c$. Thus $S(F)(c) > 0$, contradicting our assumption. You can derive a similar contradiction if F' has a negative local maximum value at $x = c$.

Examples of Functions F with $S(F)(x) \geq 0$ for some x
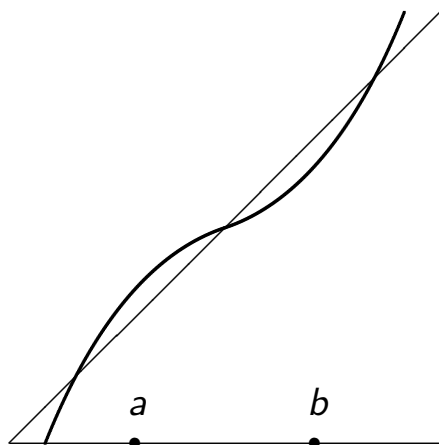
F' has a positive minimum value between $x = a$ and $x = b$, where the slopes are both 1.

F' has a negative maximum value between $x = a$ and $x = b$, where the slopes are both -1.
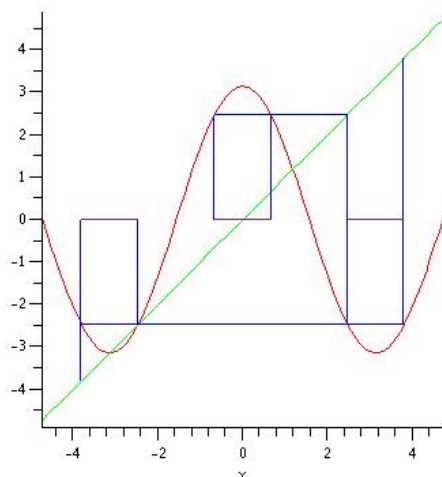
What Are Basins of Attraction?

Suppose p is an attracting fixed point for F.

Definition 1: The basin of attraction of p is the set of all points x whose orbits under F tend to p. That is,

$$\{x \in \mathbb{R} \mid \lim_{n \to \infty} F^n(x) = p\}.$$

Definition 2: The immediate basin of attraction of p is the largest interval containing p that lies in the basin of attraction.

Example 1

Consider the function

$$C(x) = \pi \cos x;$$

it has an attracting fixed point at $p = -\pi$, as you can check. Its basin of attraction is much larger than its immediate basin of attraction, as shown on the diagram to the left.

Basins of Attraction and the Schwarzian Derivative

Theorem: *Suppose $S(F)(x) < 0$ for all x. If p is an attracting periodic point for F, then either*
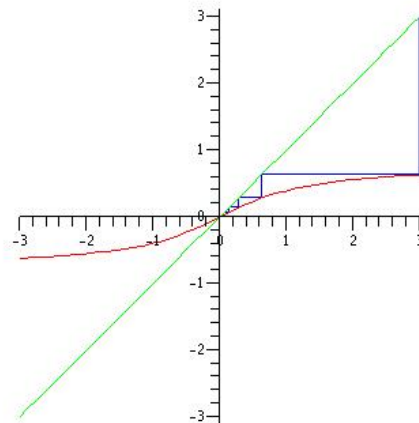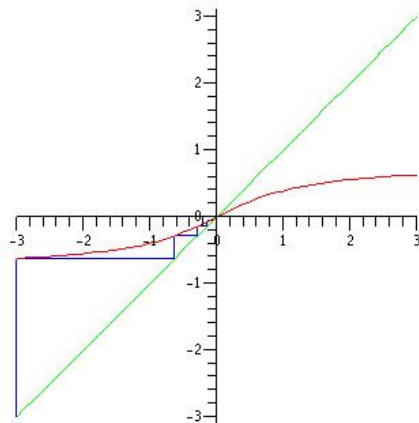
1. *the immediate basin of attraction of p extends to $+\infty$ or to $-\infty$,*
2. *or else there is a critical point of F whose orbit is attracted to the orbit of p under F.*

Note: If p is a fixed point for F, then the second option becomes

or else there is a critical point of F which is attracted to the fixed point p.

For the quadratic family, Q_c: There are no infinite basins of attraction, since for $|x| > p_+$ the orbit of x under Q_c will be unbounded. The only critical point of Q_c is $x = 0$, so if there is an attracting periodic point for Q_c, the orbit of $x = 0$ will find it.
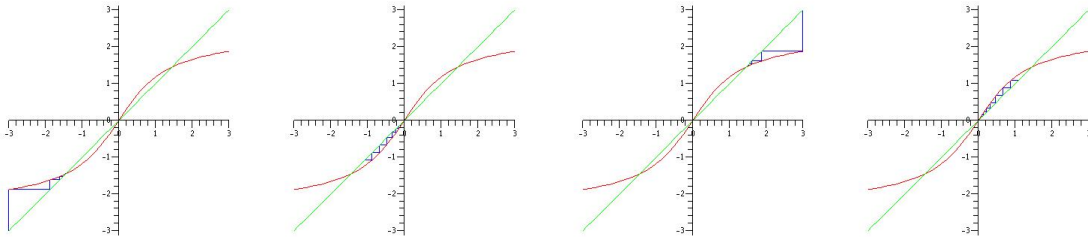
Example 2

Let $A(x) = \lambda \arctan x,\ \lambda \neq 0$. Then $A'(x) = \lambda/(1 + x^2) \neq 0$.

If $|\lambda| < 1$, then $p = 0$ is an attracting fixed point, and its immediate basin of attraction is all of $\mathbb{R}$.

Example 2, Continued

If $\lambda > 1$, then $A(x)$ has two attracting fixed points, one positive and one negative.

The immediate basin of attraction of the negative fixed point is $(-\infty, 0)$; the immediate basin of attraction of the positive fixed point is $(0, \infty)$. A has no critical points (if $\lambda \neq 0$) so all immediate basins of attractions are infinite intervals.

Example 2, Concluded

If $\lambda < -1$, then $A(x)$ has an attracting 2-cycle.
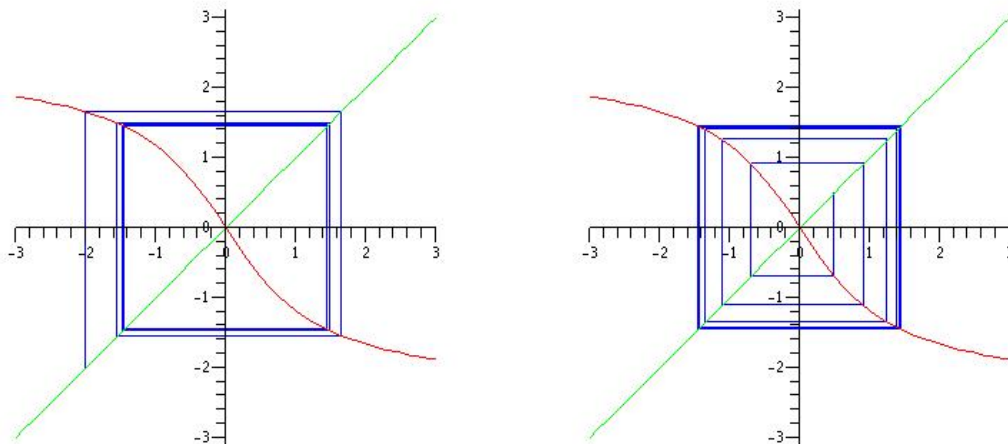
The immediate basin of attraction of the 2-cycle is all of $\mathbb{R}$; it must be infinite since A has no critical point.

Proof of the Above Theroem, If p Is a Fixed Point For F

Here is an outline, see the book for details:

1. The immediate basin of attraction of p must be an open interval.

2. If the immediate basin of attraction extends to $+\infty$ or $-\infty$ we are done.

3. Otherwise the immediate basin of attraction of p is (a, b).

4. $F : (a, b) \longrightarrow (a, b)$.

5. $F(a)$ and $F(b)$ are endpoints of $F((a, b))$.

6. There are 4 cases: $F(a) = a, F(b) = a$; $F(a) = b, F(b) = b$; or $F(a) = a, F(b) = b$; $F(a) = b, F(b) = a$.

7. In the first two cases, Rolle's Theorem implies there is a critical point in (a, b), which must be attracted to p.

For the remaining two cases, we use the fact that p is the only attracting fixed point in (a, b) and the Mean Value Theorem.

$F(a) = a, F(b) = b.$
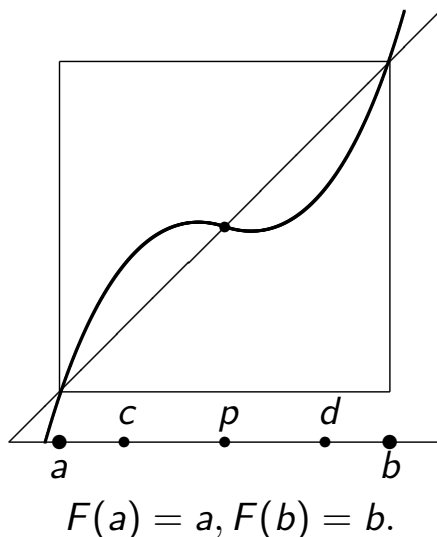
There are points c, d such that $F'(c) = 1, F'(d) = 1$. $F'(p) < 1$, but F' can't have a positive min value, so the min of F' must be negative. Then there is a critical point $x \in (a, b)$ and $F'(x) = 0$. Reduce the remaining case, $F(a) = b, F(b) = a$, to the previous case by defining $G = F^2$. Then $S(G)(x) < 0$, $G(a) = a$, $G(b) = b$, and $G(p) = p$.

Finally, $G'(x) = 0 \Rightarrow F'(F(x)) \cdot F'(x) = 0$, so either x or $F(x)$ is a critical point of F in (a, b), which completes the proof.

Newton's Method for Approximating Solutions to $F(x) = 0$
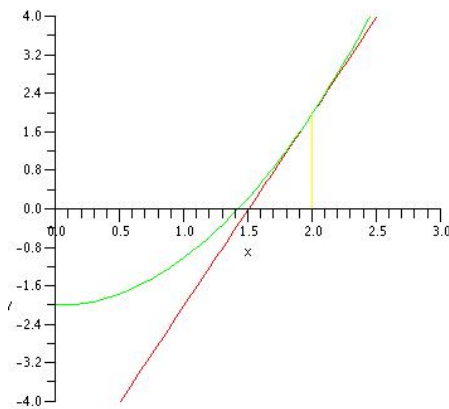
Let p be a solution to $F(x) = 0$. Suppose x_0 is an initial approximation to p. Newton's method is based on the observation that the x-intercept of the tangent line to $y = F(x)$ at $x = x_0$ is (usually) a better approximation to the value of p. The equation of the tangent is $y = F(x_0) + F'(x_0)(x - x_0)$.

Let the x-intercept of the tangent line be x_1 :

$$0 = F(x_0) + F'(x_0)(x_1 - x_0) \Leftrightarrow x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

The Newton Iteration Function

Now repeat the process to get a better approximation x_2, and so on. You can calculate a sequence of successive approximations $x_1, x_2, \ldots, x_n, \ldots$ with Newton's recursive formula:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \text{ for } n = 0, 1, 2, 3, \ldots$$

Definition: Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a function. The Newton iteration function associated to F is

$$N(x) = x - \frac{F(x)}{F'(x)}.$$

Observe: $N(p) = p \Leftrightarrow F(p) = 0$, assuming $F'(p) \neq 0$. Thus approximating solutions to an equation using Newton's method is an application of an orbit that is attracted to a fixed point.
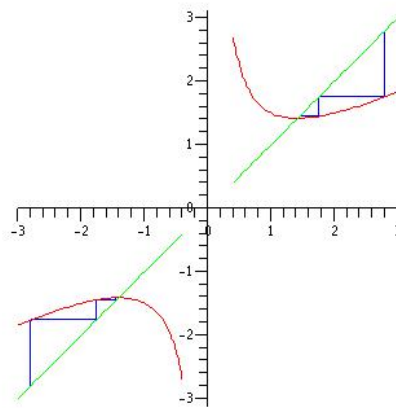
Example 1

Consider the equation

$$x^2 - 2 = 0;$$

we have $F(x) = x^2 - 2$ and

$$N(x) = x - \frac{x^2 - 2}{2x} = \frac{x}{2} + \frac{1}{x}.$$

You can see that for any $x_0 \neq 0$, the orbit of x_0 under N is attracted to one of the two fixed points of N, namely $\pm\sqrt{2}$, which are the solutions to the equation $F(x) = 0$.

Newton's Fixed Point Theorem

Definition: A root r of the equation $F(x) = 0$ has multiplicity k if $F^{(k-1)}(r) = 0$ but $F^{(k)}(r) \neq 0$. ($F^{(k)}$ is the kth derivative of F.)

> **Theorem:** *Suppose F is a function and N is the associated Newton iteration function. Then r is a root of F of multiplicity k if and only if r is a fixed point of N. Moreover, such a fixed point is always attracting.*

Proof: First suppose r has multiplicity $k = 1$; that is, $F'(r) \neq 0$. Then (as already pointed out) $F(r) = 0 \Leftrightarrow N(r) = r$. And

$$N'(r) = 1 - \frac{(F'(r))^2 - F(r)F''(r)}{(F'(r))^2} = \frac{F(r)F''(r)}{(F'(r))^2} = 0$$

since $F(r) = 0$. Consequently, r is an attracting fixed point of N.

Now suppose $k > 1$. Then there is a function $G(x)$ such that $G(r) \neq 0$ and

$$F(x) = (x - r)^k G(x).$$

Calculate:

$$F'(x) = k(x - r)^{k-1} G(x) + (x - r)^k G'(x);$$

$$F''(x) = k(k-1)(x-r)^{k-2} G(x) + 2k(x-r)^{k-1} G'(x) + (x-r)^k G''(x).$$

Consequently, after much simplification:

$$N(x) = x - \frac{(x - r) G(x)}{k G(x) + (x - r) G'(x)},$$

from which we see immediately that $N(r) = r$.

Finally, check that

$$N'(x) = \frac{k(k - 1)(G(x))^2 + 2k(x - r)G(x)G'(x) + (x - r)^2 G(x)G''(x)}{k^2(G(x))^2 + 2k(x - r)G(x)G'(x) + (x - r)^2(G'(x))^2}.$$

Since $G(r) \neq 0$,

$$N'(r) = \frac{k - 1}{k} < 1,$$

so r is indeed an attracting fixed point for N.

Note: if r is a simple root of $F(x) = 0$ then $N'(r) = 0$, so orbits that start close to r will converge to r very quickly; but if r is not a simple root of $F(x) = 0$ then the convergence to r will be slower in the sense that $N'(r)$ can be close to 1.
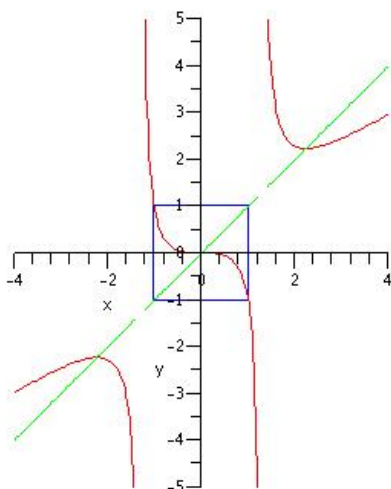
Example 1

As we know it is entirely possible for a dynamical system to have a 2-cycle. This may happen when using Newton's method to approximate a solution to $F(x) = 0$, even if there are solutions to the equation. For example:

$$F(x) = x^3 - 5x \Rightarrow N(x) = x - \frac{x^3 - 5x}{3x^2 - 5} = \frac{2x^3}{3x^2 - 5};$$

and $N(1) = -1$, $N(-1) = 1$, so $x_0 = 1$ does not result in an orbit that converges to one of the solutions to $F(x) = 0$.

Graph for Example 1

Of course Newton's method will work fine if the initial choice for x is close enough to one of the solutions to

$$x^3 - 5x = 0;$$

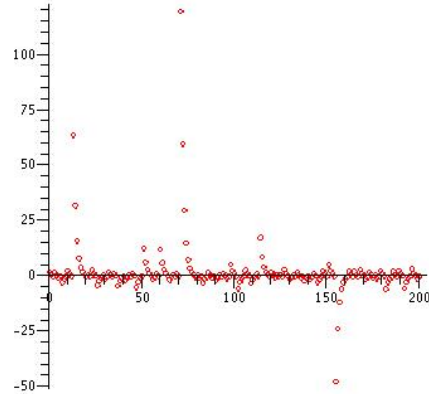or, equivalently, to one of the fixed points of $N(x)$.

Example 2

Consider the equation $x^2 + 1 = 0$ which has no real solutions.

Then

$$N(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

and the orbits of x under N go all over. The orbit to the right starts with $x = 2$ and includes 200 iterations.

In fact, $N : \mathbb{R} \longrightarrow \mathbb{R}$ is conjugate to the doubling function $D : [0, 1) \longrightarrow [0, 1)$ of Chapter 3, which is chaotic, see Chapter 10.

Another Conjugacy

The conjugacy is $C(x) = \cot(\pi x)$, for $x \in [0, 1)$.

$$
\begin{array}{ccc}
[0, 1) & \overset{D}{\longrightarrow} & [0, 1) \\
C \downarrow & & \downarrow C \\
\mathbb{R} & \underset{N}{\longrightarrow} & \mathbb{R}
\end{array}
$$

$$
\begin{aligned}
C(D(x)) &= \cot(\pi D(x)) \\
&= \cot(2\pi x) \\
&= \frac{\cos^2(\pi x) - \sin^2(\pi x)}{2\sin(\pi x)\,\cos(\pi x)} \\
&= \frac{1}{2}\left(\cot(\pi x) - \frac{1}{\cot(\pi x)}\right) \\
&= N(C(x))
\end{aligned}
$$

NB: the only fixed point of D is 0; the only 'fixed' point of N is ∞.
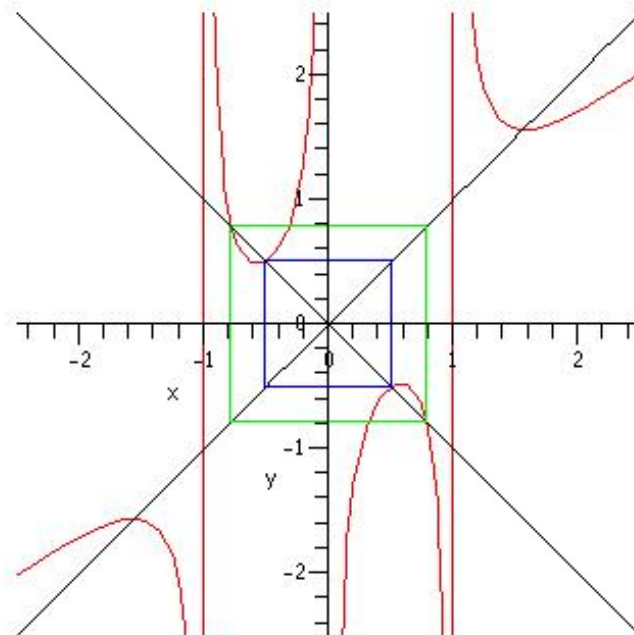
Example 3

Consider the equation $x^4 - 2x^2 - 17/16 = 0$, for which

$$F(x) = x^4 - 2x^2 - \frac{17}{16} \quad \text{and} \quad N(x) = \frac{48x^4 - 32x^2 + 17}{64(x^3 - x)}.$$

Check that

1. $x_0 = -3 \Rightarrow x_n \to -1.56081$
2. $x_0 = 0.1 \Rightarrow x_n \to -1.56081$
3. $x_0 = 1.5 \Rightarrow x_n \to 1.56081$
4. $x_0 = 0.5 \Rightarrow x_1 = -0.5 \Rightarrow x_2 = 0.5$
5. $x_0 = 0.9 \Rightarrow x_n \to -1.56081$

Graph of $N(x)$ For Example 3: $x^4 - 2x^2 - 17/16 = 0$

Some Observations for Example 3

Both fixed points of N are strongly attracting since

$$N'(x) = \frac{(x^4 - 2x^2 - 17/16)(12x^2 - 4)}{(4x^3 - 4x)^2}$$

so

$$F(x) = 0 \Rightarrow N'(x) = 0.$$

N actually has two 2-cycles:

$$N(x) = -x \Rightarrow x = \pm\frac{1}{2} \text{ or } x = \pm\frac{\sqrt{119}}{14}.$$

Only one of these 2-cycles is attracting:

$$N'(-1/2)N'(1/2) = \frac{4}{9} < 1$$

Basins of Attractions

but

$$N'(-\sqrt{119}/14)N'(\sqrt{119}/14) = \frac{2116}{121} > 1.$$

What are the basins of attractions of the attracting fixed points of N pause or of the attracting 2-cycle of N?
For Newton's method, this question is in generally very difficult to answer.
You can say that the immediate basin of attraction of the positive fixed point of N is $(1, \infty)$, and that the immediate basin of attraction of the negative fixed point of N is $(-\infty, -1)$; and you might be able to find the complete basin of attraction for each fixed point. But it would be extremely difficult to find the basin of attraction for the attracting 2-cycle of N.