MAT187H1S Lec0101 Burbulla

Chapter 13 Lecture Notes

Winter 2012

Chapter 13: Partial Derivatives
- 13.1 Functions of Two or More Variables
- 13.3 Partial Derivatives
- 13.8 Maxima and Minima of Functions of Two Variables

Introduction

Chapters 13, 14 and 15 of the text book are all concerned with multivariable calculus. How much multivariable calculus you will see in your other courses depends on your program. What we will cover in the last week of MAT 187H1S is just a quick introduction to functions of two variables,

$$z = f(x, y).$$

This will include

1. partial derivatives
2. critical points of a function of two variables
3. the second derivative test for a function of two variables

Examples

The following are all functions of two variables, x and y :

1. $z = x^2 + y^2$
2. $z = x^2 - y^2$
3. $z = x^3 + y^3 - 3xy$
4. $z = \sin(x^2 + y^2)$
5. $z = 6xy^2 - 2x^3 - 3y^4$
6. $z = e^{-(x^2+y^2)}$

The graph of a function $z = f(x, y)$ is the set of all points (x, y, z) which satisfy the equation $z = f(x, y)$. The graph is called a surface. Another way to represent $z = f(x, y)$ graphically is by a contour plot, which is a collection of curves in the plane joining all points (x, y) with the same z value.

Example 1: $z = x^2 + y^2$

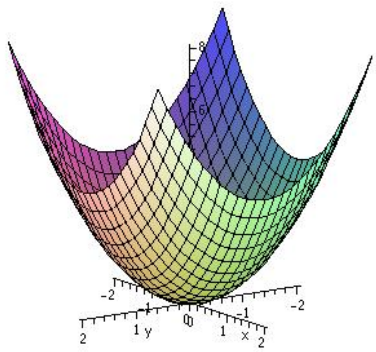


Figure: 3D Plot

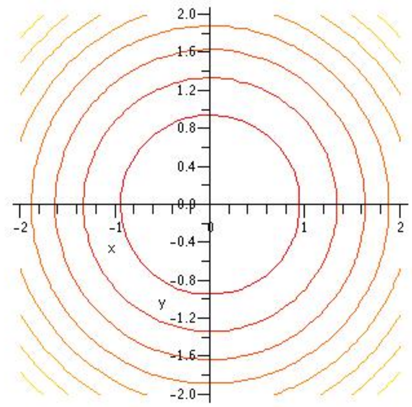


Figure: Contour Plot

Example 2: $z = x^2 - y^2$

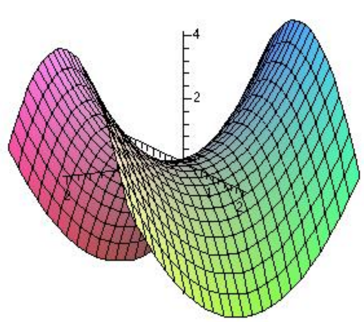


Figure: 3D Plot

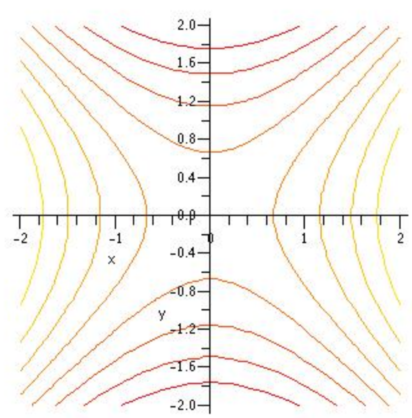


Figure: Contour Plot

Example 3: $z = x^3 + y^3 - 3xy$

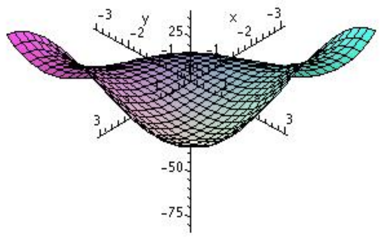


Figure: 3D Plot

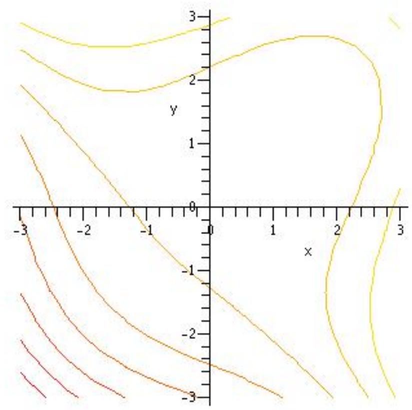


Figure: Contour Plot

Example 4: $z = \sin(x^2 + y^2)$

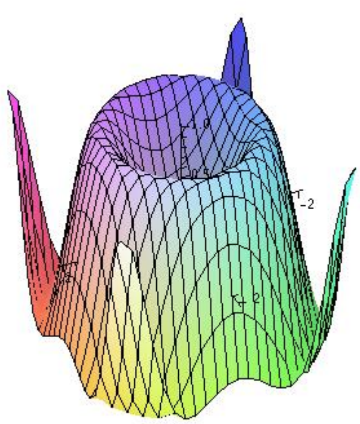


Figure: 3D Plot

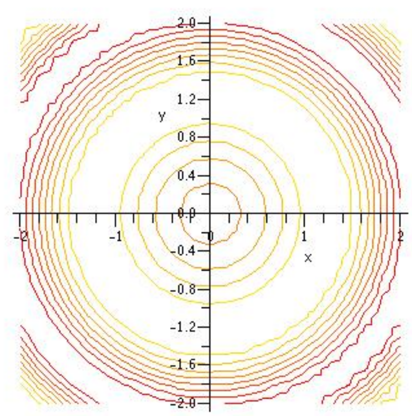


Figure: Contour Plot

Example 5: $z = 6xy^2 - 2x^3 - 3y^4$

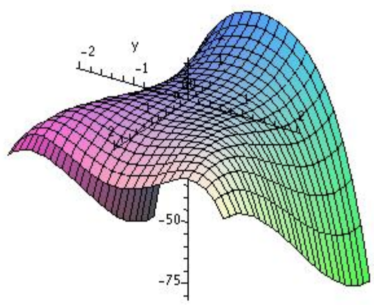


Figure: 3D Plot

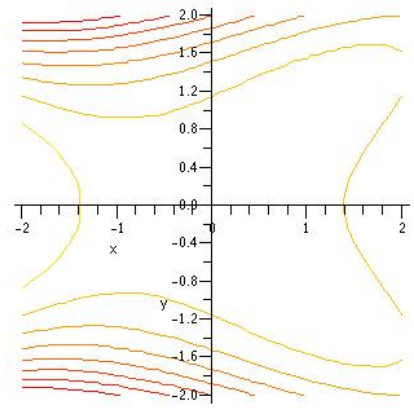


Figure: Contour Plot

Example 6: $z = e^{-(x^2+y^2)}$

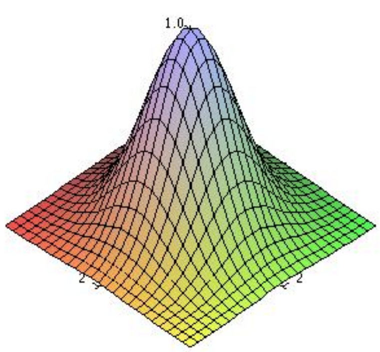


Figure: 3D Plot

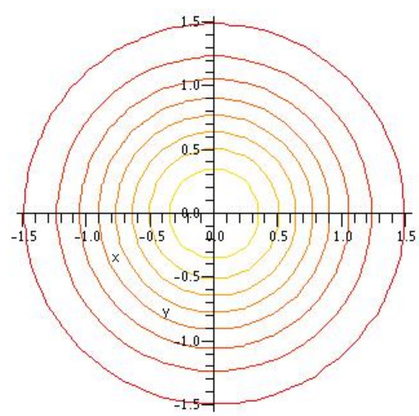


Figure: Contour Plot

What is a Partial Derivative?

If $z = f(x, y)$ is a function of two variables, then there are two possible derivatives: these two derivatives are called partial derivatives.

1. $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
   is called the partial derivative of z with respect to x.
2. $$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$
   is called the partial derivative of z with respect to y.

Alternate notations: f_x or z_x or $D_x z$, f_y or z_y or $D_y z$.

How To Calculate Partial Derivatives

Calculating parital derivatives is easy, in the sense that you can use the same derivative formulas and techniques you already know. All you do, to calculate a partial derivative with respect to one variable, is treat the other variable(s) as a constant. For example, if $z = x^2 + y^2$, then

$$\frac{\partial z}{\partial x} = 2x \text{ and } \frac{\partial z}{\partial y} = 2y.$$

Of course, things can get more complicated!

Example 1

1. Let $z = x^3 + y^3 - 3xy$, then
$$\frac{\partial z}{\partial x} = 3x^2 - 3y \text{ and } \frac{\partial z}{\partial y} = 3y^2 - 3x.$$
2. Let $z = 6xy^2 - 2x^3 - 3y^4$, then
$$\frac{\partial z}{\partial x} = 6y^2 - 6x^2 \text{ and } \frac{\partial z}{\partial y} = 12xy - 12y^3.$$
3. Let $z = \sin(x^2 + y^2)$, then
$$\frac{\partial z}{\partial x} = 2x\cos(x^2 + y^2) \text{ and } \frac{\partial z}{\partial y} = 2y\cos(x^2 + y^2).$$

Geometric Interpretation of Partial Derivatives

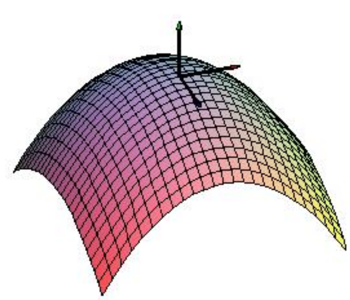
Suppose $f_x(a,b), f_y(a,b)$ are the partial derivatives of f at (a,b).

Then $\mathbf{u} = < 1, 0, f_x(a,b) >$ is a tangent vector to the curve with parametric equations,

$$x = t, y = b, z = f(t,b).$$

Similarly, $\mathbf{v} = < 0, 1, f_y(a,b) >$ is a tangent vector to the curve with parametric equations,

$$x = a, y = t, z = f(a,t).$$



Then $\mathbf{n} = \mathbf{v} \times \mathbf{u} = < f_x(a,b), f_y(a,b), -1 >$ is a normal vector.

Tangent Plane to a Surface $z = f(x, y)$ at $(x, y) = (a, b)$

For a surface $z = f(x, y)$ there are many tangent lines passing through the point $(a, b, f(a, b))$, but only one tangent plane. Its equation is given by

$$\mathbf{n}\cdot < x - a, y - b, z - f(a, b) >= 0,$$

with

$$\mathbf{n} =< 0, 1, f_y(a, b) > \times < 1, 0, f_x(a, b) > \; =< f_x(a, b), f_y(a, b), -1 > .$$

Thus the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$x\, f_x(a, b) + y\, f_y(a, b) - z = a\, f_x(a, b) + b\, f_y(a, b) - f(a, b).$$

Example 2

Let $z = x^3 + y^3 - 3xy$, for which

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \text{ and } \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

At the point $(x, y) = (1, 3)$, we have $z = 19$ and

$$\frac{\partial z}{\partial x} = -6 \text{ and } \frac{\partial z}{\partial y} = 24.$$

So the equation of the tangent plane to the surface at $(1, 3, 19)$ is

$$-6x + 24y - z = -6 + 72 - 19 \Leftrightarrow 6x - 24y + z = -47.$$

Second Order Partial Derivatives

Since each partial derivative

$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}$$

is itself a function of x and y, each can be differentiated again, either with respect to x or with respect to y. This gives four possible second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Example 3

Let $z = x^3 + y^3 - 3xy$, for which

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \text{ and } \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

Then

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y\right) = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3y^2 - 3x\right) = -3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2 - 3y\right) = -3 \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(3y^2 - 3x\right) = 6y.$$

It is usually true that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y};$$

always, for the functions you will see in MAT187H1S.

Example 4

Let $z = 6xy^2 - 2x^3 - 3y^4$, for which

$$\frac{\partial z}{\partial x} = 6y^2 - 6x^2 \text{ and } \frac{\partial z}{\partial y} = 12xy - 12y^3.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(6y^2 - 6x^2\right) = -12x,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(12xy - 12y^3\right) = 12y,$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(6y^2 - 6x^2\right) = 12y,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(12xy - 12y^3\right) = 12x - 36y^2.$$

Example 5

Let $z = \sin(x^2 + y^2)$, for which

$$\frac{\partial z}{\partial x} = 2x\cos(x^2 + y^2) \text{ and } \frac{\partial z}{\partial y} = 2y\cos(x^2 + y^2).$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(2x\cos(x^2 + y^2)\right) = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2),$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2y\cos(x^2 + y^2)\right) = -4\,y\,x\,\sin(x^2 + y^2),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2x\cos(x^2 + y^2)\right) = -4\,x\,y\,\sin(x^2 + y^2),$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(2y\cos(x^2 + y^2)\right) = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2).$$

Extreme Values of a Function of Two Variables

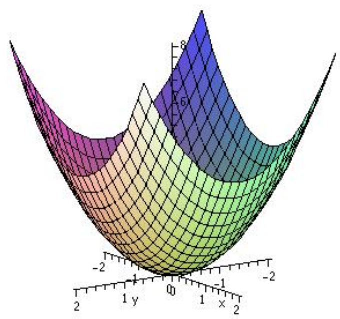


Figure: Minimum point.

$$f(a,b) = m \leq f(x,y)$$

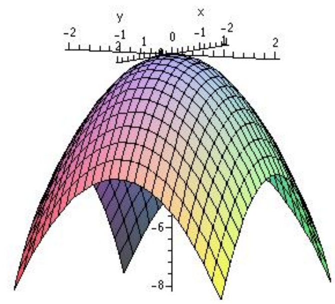


Figure: Maximum point.

$$f(x,y) \leq M = f(a,b)$$

Critical Points of a Function of Two Variables

Definition: $(x,y) = (a,b)$ is a critical point of $z = f(x,y)$, if

$$f_x(a,b) = 0 \text{ or is undefined, and } f_y(a,b) = 0 \text{ or is undefined.}$$

Since this is just a quick introduction to multivariable calculus, we will consider only critical points for which

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0.$$

If (a,b) is a maximum or minimum point of $z = f(x,y)$, then the normal vector to the surface at $(x,y) = (a,b)$ must be parallel to the z-axis; that is, $f_x(a,b) = 0$ and $f_y(a,b) = 0$. However, this is only a necessary condition; not a sufficient condition. That is, there are critical points for which f has neither a maximum nor a minimum value.

Saddle Points

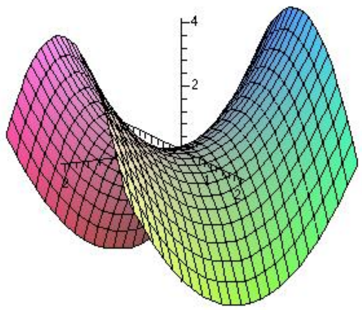


Figure: A saddle point.

A critical point that is neither a maximum nor a minimum point, is called a saddle point. An example is $z = x^2 - y^2$ at $(x, y) = (0, 0)$. At this point both partial derivatives of z are zero, but $z = 0$ is neither a max nor a min:

$$x = 0 \Rightarrow z = -y^2 \leq 0;$$

$$y = 0 \Rightarrow z = x^2 \geq 0.$$

Example 1

Let $z = x^3 + y^3 - 3xy$, for which

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \text{ and } \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

$$\begin{cases} 3x^2 - 3y & = & 0 \\ 3y^2 - 3x & = & 0 \end{cases} \Leftrightarrow \begin{cases} x^2 & = & y \\ y^2 & = & x \end{cases} \Rightarrow x^4 = x \Rightarrow x = 0 \text{ or } 1.$$

Since $y = x^2$, there are only two critical points:

$$(x, y) = (0, 0), \text{ or } (1, 1).$$

$(0, 0, 0)$ is a saddle point since $x = 0 \Rightarrow z = y^3$, for which $z = 0$ is neither a max nor a min.
At $(x, y) = (1, 1), z = -1$; is it a max or a min?

Example 2

Let $z = 6xy^2 - 2x^3 - 3y^4$, for which

$$\frac{\partial z}{\partial x} = 6y^2 - 6x^2 \text{ and } \frac{\partial z}{\partial y} = 12xy - 12y^3.$$

$$\left\{ \begin{array}{rcl} 6y^2 - 6x^2 & = & 0 \\ 12xy - 12y^3 & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} y^2 & = & x^2 \\ xy & = & y^3 \end{array} \right. \Rightarrow y = 0 \text{ or } x = y^2$$

$$\Rightarrow \quad x = 0 \text{ or } x = x^2 \Leftrightarrow x = 0 \text{ or } x = 1$$

$$\Rightarrow \quad (x, y) = (0, 0) \text{ or } (1, \pm 1), \text{ since } y^2 = x.$$

$(0, 0, 0)$ is a saddle point, since $y = 0 \Rightarrow z = -2x^3$, which has neither a max nor a min at $x = 0$.
What about $z = 1$; max or min at $(1, \pm 1)$?

Example 3

Let $z = e^{-(x^2+y^2)}$, for which

$$\frac{\partial z}{\partial x} = -2xe^{-(x^2+y^2)} \text{ and } \frac{\partial z}{\partial y} = -2ye^{-(x^2+y^2)}.$$

The only critical point is $(x, y) = (0, 0)$, at which $z = 1$. Is $z = 1$ a maximum or minimum value? Its a maximum value, since

$$x^2 + y^2 \geq 0 \quad \Rightarrow \quad -(x^2 + y^2) \leq 0$$

$$\Rightarrow \quad e^{-(x^2+y^2)} \leq e^0 = 1$$

Second Derivative Test for $z = f(x, y)$

Let

$$\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 z}{\partial y \partial x} = \det \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix};$$

let $(x, y) = (a, b)$ be a critical point of f such that both

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

1. if $\Delta > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ at (a, b), then $f(a, b)$ is a minimum.
2. if $\Delta > 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$ at (a, b), then $f(a, b)$ is a maximum.
3. if $\Delta < 0$ at (a, b), then $(a, b, f(a, b))$ is a saddle point.
4. if $\Delta = 0$ at (a, b), this test fails.

Comments About the Test

1. This test does not apply to critical points for which

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial z}{\partial y}$$

is undefined.

2. $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \Rightarrow \Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

3. $$\Delta > 0 \Rightarrow \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \geq 0 \Rightarrow \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} > 0$$

This means that $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ have the same signs.

Examples: $z = x^2 + y^2$ and $z = x^2 - y^2$

1. Let $z = x^2 + y^2$:

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = 2y; \frac{\partial^2 z}{\partial x^2} = 2, \frac{\partial^2 z}{\partial y^2} = 2, \frac{\partial^2 z}{\partial x \partial y} = 0; \Delta = 4.$$

The only critical point is $(x, y) = (0, 0)$; $z = 0$ is a minimum value, since $\Delta > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$.

2. Let $z = x^2 - y^2$:

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = -2y; \frac{\partial^2 z}{\partial x^2} = 2, \frac{\partial^2 z}{\partial y^2} = -2, \frac{\partial^2 z}{\partial x \partial y} = 0; \Delta = -4.$$

The only critical point is $(x, y) = (0, 0)$; and $(0, 0, 0)$ is a saddle point, since $\Delta < 0$.

Example 1, Revisited: $z = x^3 + y^3 - 3xy$

We have

$$\frac{\partial z}{\partial x} = 3x^2 - 3y, \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

Critical points:

$$\begin{cases} 3x^2 - 3y & = & 0 \\ 3y^2 - 3x & = & 0 \end{cases} \Leftrightarrow \begin{cases} x^2 & = & y \\ y^2 & = & x \end{cases} \Leftrightarrow (x, y) = (0, 0) \text{ or } (1, 1).$$

$$\frac{\partial^2 z}{\partial x^2} = 6x, \frac{\partial^2 z}{\partial y^2} = 6y, \frac{\partial^2 z}{\partial x \partial y} = -3; \Delta = 36xy - (-3)^2 = 36xy - 9.$$

At $(0, 0), \Delta = -9 < 0$, so there is a saddle point at $(0, 0, 0)$; at $(1, 1), \Delta = 27 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 6 > 0$, so $z = -1$ is a minimum value.

Example 2, Revisited: $z = 6xy^2 - 2x^3 - 3y^4$

$$\frac{\partial z}{\partial x} = 6y^2 - 6x^2, \frac{\partial z}{\partial y} = 12xy - 12y^3.$$

Critical points:

$$\begin{cases} 6y^2 - 6x^2 = 0 \\ 12xy - 12y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} y^2 = x^2 \\ xy = y^3 \end{cases}$$

$$\Rightarrow y = 0 \text{ or } x = y^2$$

$$\Rightarrow x = 0 \text{ or } x = x^2 \Leftrightarrow x = 0 \text{ or } x = 1$$

$$\Rightarrow (x, y) = (0, 0) \text{ or } (1, \pm 1), \text{ since } y^2 = x.$$

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Example 2, Continued

$$\frac{\partial^2 z}{\partial x^2} = -12x, \frac{\partial^2 z}{\partial y^2} = 12x - 36y^2, \frac{\partial^2 z}{\partial x \partial y} = 12y;$$

so

$$\Delta = (-12x)(12x - 36y^2) - 144y^2.$$

At $(1, \pm 1)$,

$$\Delta = 144 > 0, \text{ and } \frac{\partial^2 z}{\partial x^2} = -12 < 0,$$

so $z = 1$ is maximum value, at $(x, y) = (1, \pm 1)$. At $(0, 0), \Delta = 0$, so the test fails. For $y = 0 : z = -2x^3$. Note: $z > 0$ if $x < 0$ and $z < 0$ if $x > 0$. So $(0, 0, 0)$ is a saddle point.

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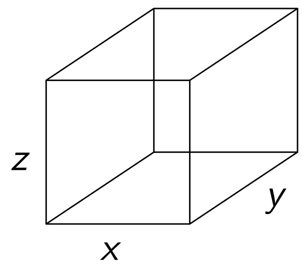
Example 3

Find the dimensions of an open-topped box with volume 4000 cm^3 that has minimum surface area.



Let the dimensions of the box be $x \times y \times z$. The volume of the box is $V = xyz$. We have

$$V = 4000 \Leftrightarrow z = \frac{4000}{xy}.$$

The surface area (sans top) is

$$SA = xy + 2yz + 2xz.$$

$$z = \frac{4000}{xy} \Rightarrow SA = xy + \frac{8000}{x} + \frac{8000}{y}.$$

Example 3, Continued

$$\frac{\partial SA}{\partial x} = y - \frac{8000}{x^2}; \frac{\partial SA}{\partial y} = x - \frac{8000}{y^2}.$$

$$\begin{aligned}
\frac{\partial SA}{\partial x} = 0 \text{ and } \frac{\partial SA}{\partial y} = 0 \quad &\Rightarrow \quad y - \frac{8000}{x^2} = 0 \text{ and } x - \frac{8000}{y^2} = 0 \\
&\Rightarrow \quad x^2 y = 8000 = xy^2 \\
&\Rightarrow \quad y = x, \text{ since } x > 0, y > 0 \\
&\Rightarrow \quad x^3 = 8000 \Rightarrow x = 20 \text{ and } y = 20.
\end{aligned}$$

$$\frac{\partial^2 SA}{\partial x^2} = \frac{16000}{x^3}; \frac{\partial^2 SA}{\partial y^2} = \frac{16000}{y^3}; \frac{\partial^2 SA}{\partial x \partial y} = 1; \Delta = \frac{16000^2}{x^3 y^3} - 1.$$

At $x = 20, y = 20, z = 10$, $\Delta = 3 > 0$ and $\frac{\partial^2 SA}{\partial x^2} = 2 > 0$, so the surface area has been minimized.

Extreme Values on a Closed and Bounded Domain

If the domain of the function $f(x, y)$ is a closed and bounded subset R of $\mathbb{R}^2$, then there will be points (a, b) and (c, d) in R such that

$$m = f(a, b) \leq f(x, y) \leq f(c, d) = M,$$

for all $(x, y) \in R$. m and M are called the extreme values of f. How to find them? The extreme values of f on the closed and bounded subset R will occur at

1. a critical point of f in R, or at
2. a point on the boundary of R.

We will not go into all the detail of what a 'closed and bounded' set is, or what the boundary of a region R is; but two examples will illustrate the basic ideas.

Example 4

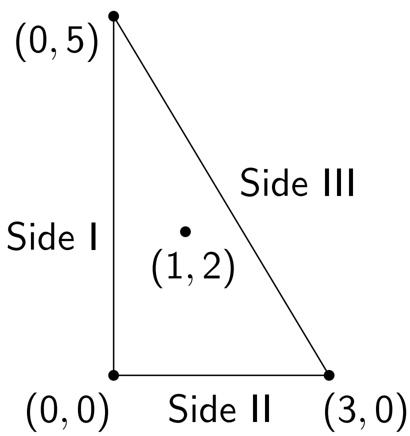
Find the absolute extrema of $f(x, y) = x^2 + 2y^2 - x$ on the disc $R = \{(x, y) \mid x^2 + y^2 \leq 4\}$. **Solution:** The only critical point of f is $(x, y) = (1/2, 0)$, as you can check, and is in R. The boundary of R is the circle with equation $x^2 + y^2 = 4$. On the boundary

$$y^2 = 4 - x^2, \text{ for } -2 \leq x \leq 2,$$

and $z = f(x) = x^2 + 2(4 - x^2) - x = 8 - x - x^2$. This function, restricted to the interal $[-2, 2]$, has one critical point at $x = -1/2$, for which $y = \pm\sqrt{15}/2$, and two endpoints at $x = \pm 2$, for which $y = 0$. There are five possible points in R at which the extrema of f could occur: $(\pm 2, 0)$, $(-1/2, \pm\sqrt{15}/2)$ and $(1/2, 0)$. Comparing z values: the minimum value of f on R is $m = -0.25$ at $(1/2, 0)$; and the maximum of f on R is $M = 8.25$ at $(-1/2, \sqrt{15}/2)$.

Example 5

Find the absolute extrema of $f(x, y) = 3xy - 6x - 3y + 7$ on the region R which is inside and including the triangle with vertices $(0, 0)$, $(3, 0)$ and $(0, 5)$. **Solution:**



The boundary of R is the triangle, which has three sides:

1. Side I: $x = 0$, for $0 \le y \le 5$.
2. Side II: $y = 0$, for $0 \le x \le 3$.
3. Side III: $y = 5 - \frac{5}{3}x$, for $0 \le x \le 3$.

Also: check that the only critical point of f is $(x, y) = (1, 2)$, which is inside the triangle. At this critical point $z = f(1, 2) = 1$.

Example 5, Continued

On Side I: $z = f(0, y) = -3y + 7$, for $0 \le y \le 5$. This function has no critical points, just two endpoints: $y = 0$ and $y = 5$.
On Side II: $z = f(x, 0) = -6x + 7$, for $0 \le x \le 3$. This function has no critical points, just two endpoints: $x = 0$ and $x = 3$.
On Side III:

$$z = 3x\left(5 - \frac{5}{3}x\right) - 6x - 3\left(5 - \frac{5}{3}x\right) + 7 = 14x - 5x^2 - 8,$$

which has a critcal point at $x = 7/5$. Compare z values at the following five points in $R: (0, 0), (3, 0), (0, 5), (1, 2)$ and $\left(\frac{7}{5}, \frac{8}{3}\right)$ to find that the minimum value of f on R is $m = -11$ at $(3, 0)$; and the maximum value of f on R is $M = 7$ at $(0, 0)$.