MAT187H1F Lec0101 Burbulla

Chapter 12 Lecture Notes

Spring 2017

Vector Operations in $\mathbb{R}^3$: the Dot and Cross Products

There are two important operations on vectors in $\mathbb{R}^3$: the dot product and the cross product. Note that in Briggs, vectors are written as row vectors. For

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

we define

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

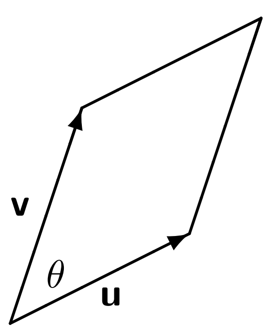
and

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - v_2 u_3, u_3 v_1 - v_3 u_1, u_1 v_2 - v_1 u_2 \rangle.$$

The length of a vector, in Briggs, is represented by absolute value signs:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Geometric Properties of $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$



Let $\theta \in [0, \pi]$ be the angle between the two vectors $\mathbf{u}, \mathbf{v}$.

1. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ and $|\mathbf{u}| = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$.
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ but $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$.
4. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$.
5. $\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$.
6. $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.
7. The area of the parallelogram spanned by the non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$A = |\mathbf{u} \times \mathbf{v}|.$$

Lines and Planes in $\mathbb{R}^3$.

1. If $\mathbf{d} \neq \mathbf{0}$, $\mathbf{x} = t\,\mathbf{d}$ is the vector equation of a line parallel to the direction vector $\mathbf{d}$ and passing through the origin.
2. If $\mathbf{d} \neq \mathbf{0}$, $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{d}$ is the vector equation of a line parallel to the direction vector $\mathbf{d}$ and passing through the point $\mathbf{x}_0$.
3. If $\mathbf{u}, \mathbf{v}$ are independent vectors then $\mathbf{x} = s\,\mathbf{u} + t\,\mathbf{v}$ is the vector equation of the plane parallel to both direction vectors $\mathbf{u}, \mathbf{v}$ and passing through the origin.
4. If $\mathbf{u}, \mathbf{v}$ are independent vectors then $\mathbf{x} = \mathbf{x}_0 + s\,\mathbf{u} + t\,\mathbf{v}$ is the vector equation of the plane parallel to both direction vectors $\mathbf{u}, \mathbf{v}$ and passing through the point $\mathbf{x}_0$. If $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, then the point-normal equation of this plane is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.

Example 1

Find the point-normal equation of the plane with vector equation

$$\langle x, y, z \rangle = \langle 1, -2, 0 \rangle + s\langle 1, 0, -2 \rangle + t\langle 0, 2, 1 \rangle.$$

Solution: the normal vector $\mathbf{n}$ to the plane is

$$\mathbf{n} = \langle 1, 0, -2 \rangle \times \langle 0, 2, 1 \rangle = \langle 4, -1, 2 \rangle.$$

So the equation of the plane is

$$\langle 4, -1, 2 \rangle \cdot \langle x - 1, y + 2, z \rangle = 0 \Leftrightarrow 4x - y + 2z = 6.$$

Curves in Space

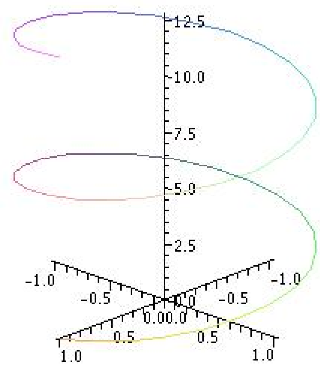
If x, y and z are all functions of t, then $\mathbf{r}(t) = \langle x, y, z \rangle$ is called a vector-valued function and the graph of $\mathbf{r}(t)$ will be a curve in 3-dimensional space. Here are two examples:

Example 2:

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

describes a helix. The parametric equations of the helix are

$$x = \cos t, y = \sin t, z = t.$$



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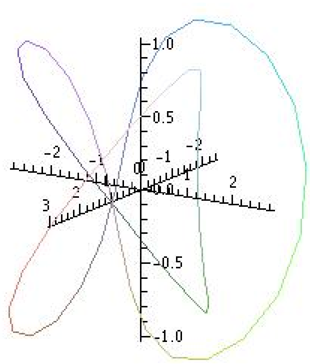
Example 3: a Knot

$$\mathbf{r}(t) = \langle (2 + \cos(3t/2)) \cos(t), (2 + \cos(3t/2)) \sin(t), \sin(3t/2) \rangle$$

is the vector equation of a knot. Its parametric equations are



$$x = (2 + \cos(3t/2)) \cos(t),$$

$$y = (2 + \cos(3t/2)) \sin(t),$$

$$z = \sin(3t/2).$$

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Different Descriptions of Curves in Space

Let

$$\mathbf{i} =< 1, 0, 0 >, \mathbf{j} =< 0, 1, 0 >, \mathbf{k} =< 0, 0, 1 >$$

be the standard basis of unit vectors in $\mathbb{R}^3$. If x, y and z are functions of t, then

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} =< x, y, z >$$

is called a vector-valued function, or the position vector of a parametric curve in space. If $z = 0$, then

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} =< x, y >$$

is the position vector of a parametric curve in the plane. With vectors we can include both cases in the same formula.

Orientation of Curves

A curve in space described by

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} =< x, y, z >$$

for $a \leq t \leq b$ is more than just a set of points; the curve also includes an orientation. The positive or forward direction along the curve is the direction in which the curve is generated as the parameter increases from $t = a$ to $t = b$. The opposite orientation is called the negative or backward direction. The same definitions apply to curves in two dimensions. So for example, the positive direction along the circle

$$\mathbf{r}(t) =< \cos t, \sin t >, \text{ for } 0 \leq t \leq 2\pi$$

is counterclockwise.

Limits and Continuity for Vector-Valued Functions

The limit of a vector-valued function is defined componentwise:

$$\lim_{t\to t_0} \mathbf{r}(t) = \left(\lim_{t\to t_0}\right) x\,\mathbf{i} + \left(\lim_{t\to t_0} y\right)\mathbf{j} + \left(\lim_{t\to t_0} z\right)\mathbf{k}.$$

Thus calculating the limit of a vector-valued functions is "three limits in one." And, as you may expect, the vector valued function is continuous at $t = t_0$ if

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

This means that the vector-valued function $\mathbf{r}(t)$ is continuous at $t = t_0$ if and only if its component functions, x, y and z, are all continuous at $t = t_0$.

Example 4

Let

$$\mathbf{r}(t) = \frac{\sin(2t)}{t}\,\mathbf{i} + \frac{t+1}{t^2-1}\,\mathbf{j} + \ln|t+2|\,\mathbf{k}.$$

Find all the discontinuities of $\mathbf{r}$. Which are removable?

Solution: the discontinuities of $\mathbf{r}$ are at $t = 0$, $t = \pm 1$ and $t = -2$. The discontinuities at $t = 1$ and $t = -2$ are not removable, but the ones at $t = 0$ and $t = -1$ are. Why? To make $\mathbf{r}$ continuous at these two points define

$$\mathbf{r}(0) = 2\,\mathbf{i} - \,\mathbf{j} + \ln 2\,\mathbf{k} \text{ and } \mathbf{r}(-1) = \sin 2\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}.$$

Derivatives and Integrals of Vector Valued Functions

Suppose

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

is a vector-valued function. Just as with limits, we define derivatives and integrals of $\mathbf{r}(t)$ componentwise:

1. $$\frac{d\mathbf{r}(t)}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$
2. $$\int \mathbf{r}(t)\,dt = \int x\,dt\,\mathbf{i} + \int y\,dt\,\mathbf{j} + \int z\,dt\,\mathbf{k}$$

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Example 1

Let

$$\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + \ln t\,\mathbf{k}.$$

Then

$$\frac{d\mathbf{r}(t)}{dt} = \frac{de^t}{dt}\,\mathbf{i} + \frac{d\sqrt{t}}{dt}\,\mathbf{j} + \frac{d\ln t}{dt}\,\mathbf{k} = e^t\,\mathbf{i} + \frac{1}{2\sqrt{t}}\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}.$$

Similarly,

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \int e^t\,dt\,\mathbf{i} + \int \sqrt{t}\,dt\,\mathbf{j} + \int \ln t\,dt\,\mathbf{k} \\ &= e^t\,\mathbf{i} + \frac{2}{3}t^{3/2}\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}.\end{aligned}$$

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Linearity of Differentiation and Integration

Let **r** and **s** be two vector-valued functions of t; let k be a scalar.

1. $$\frac{d(\mathbf{r} \pm \mathbf{s})}{dt} = \frac{d\mathbf{r}}{dt} \pm \frac{d\mathbf{s}}{dt}$$
2. $$\frac{d(k\mathbf{r})}{dt} = k\frac{d\mathbf{r}}{dt}$$
3. $$\int (\mathbf{r} \pm \mathbf{s})\, dt = \int \mathbf{r}\, dt \pm \int \mathbf{s}\, dt$$
4. $$\int (k\mathbf{r})\, dt = k \int \mathbf{r}\, dt$$

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Three Product Rules

Let **r** and **s** be two vector-valued functions of t in $\mathbb{R}^3$; let $f(t)$ be a scalar-valued function. There are three possible products with three corresponding product rules:

1. $$\frac{d(f(t)\,\mathbf{r})}{dt} = f(t)\frac{d\mathbf{r}}{dt} + \mathbf{r}\frac{df(t)}{dt}$$
2. $$\frac{d(\mathbf{r} \cdot \mathbf{s})}{dt} = \mathbf{s} \cdot \frac{d\mathbf{r}}{dt} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt}$$
3. $$\frac{d(\mathbf{r} \times \mathbf{s})}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}$$

Note: rules 1. and 2. apply as well if **r** and **s** are vectors in $\mathbb{R}^2$.

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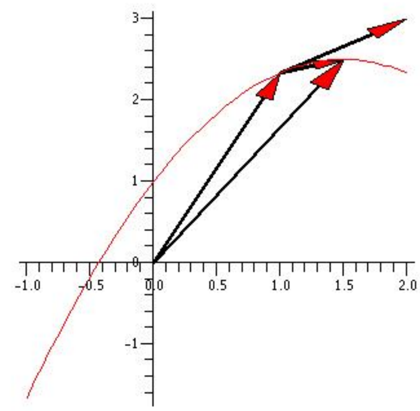
Tangent Vectors; Unit Tangent Vectors

Let $\mathbf{r}$ be a position vector. Then

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t)$$

is a tangent vector to the curve, as illustrated by a geometric interpretation of the vector calculation

$$\frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$



The vector $T(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is called the unit tangent vector.

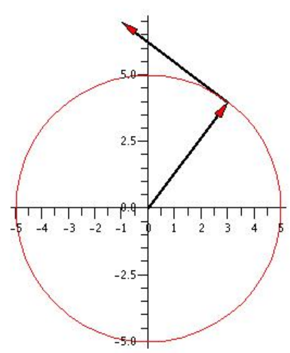
Example 2

If $\mathbf{r}$ is the position vector of a circle with radius a in the plane, then $\mathbf{r}\cdot\mathbf{r} = a^2$. Consequently:

$$\frac{d(\mathbf{r}\cdot\mathbf{r})}{dt} = 0$$

$$\Rightarrow \quad 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$$

$$\Rightarrow \quad \mathbf{r} \perp \frac{d\mathbf{r}}{dt}$$



That is: the tangent vector is always perpendicular to the position vector. Note: we didn't even use the components of $\mathbf{r}$.

Example 3

For $t > 0$ let

$$\mathbf{r}(t) = t^2\,\mathbf{i} + 4t\,\mathbf{j} + 4\ln t\,\mathbf{k}.$$

Find the unit tangent vector to $\mathbf{r}$ at t.

Solution:

$$\begin{aligned}\mathbf{r}'(t) &= \frac{dt^2}{dt}\,\mathbf{i} + \frac{d(4t)}{dt}\,\mathbf{j} + \frac{d(4\ln t)}{dt}\,\mathbf{k} \\ &= 2t\,\mathbf{i} + 4\,\mathbf{j} + \frac{4}{t}\,\mathbf{k}; \\ |\mathbf{r}'(t)| &= \sqrt{(2t)^2 + 4^2 + \left(\frac{4}{t}\right)^2}\end{aligned}$$

$$\begin{aligned}&= \sqrt{4t^2 + 16 + \frac{16}{t^2}} \\ &= 2\sqrt{t^2 + 4 + \frac{4}{t^2}} \\ &= 2\sqrt{\left(t + \frac{2}{t}\right)^2} \\ &= 2\left(t + \frac{2}{t}\right), \text{ since } t > 0.\end{aligned}$$

Thus the unit tangent vector at t is

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{t\,\mathbf{i} + 2\,\mathbf{j} + \frac{2}{t}\,\mathbf{k}}{t + 2/t} = \frac{t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2\,\mathbf{k}}{t^2 + 2}$$

Motion in Space

If $\mathbf{r}$ is the position vector of a particle at time t, then

1. the velocity of the particle at time t is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

2. the speed of the particle at time t is

$$|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

3. the acceleration of the particle at time t is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

Example 1

Find the velocity and position of a particle at time t if

$$\mathbf{a} = 2\,\mathbf{i} + 6t\,\mathbf{k};\ \mathbf{v}_0 = \mathbf{i} - \mathbf{j};\ \mathbf{r}_0 = \mathbf{j} - \mathbf{k}.$$

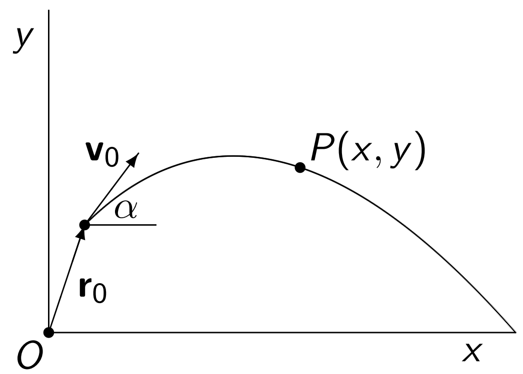
Solution: $\mathbf{v} = \int \mathbf{a}\,dt = 2t\,\mathbf{i} + 3t^2\,\mathbf{k} + \mathbf{C}$. Use initial velocity at $t = 0$ to find that $\mathbf{C} = \mathbf{i} - \mathbf{j}$. Then

$$\mathbf{r} = \int \mathbf{v}\,dt = \int \left((2t+1)\,\mathbf{i} - \mathbf{j} + 3t^2\,\mathbf{k}\right)\,dt = (t^2+t)\,\mathbf{i} - t\,\mathbf{j} + t^3\,\mathbf{k} + \mathbf{D}$$

Use the initial position at $t = 0$ to find that $\mathbf{D} = \mathbf{j} - \mathbf{k}$. Thus

$$\mathbf{r} = (t^2+t)\,\mathbf{i} + (1-t)\,\mathbf{j} + (t^3-1)\,\mathbf{k}.$$

Motion Due to Gravity, Without Air Resistance

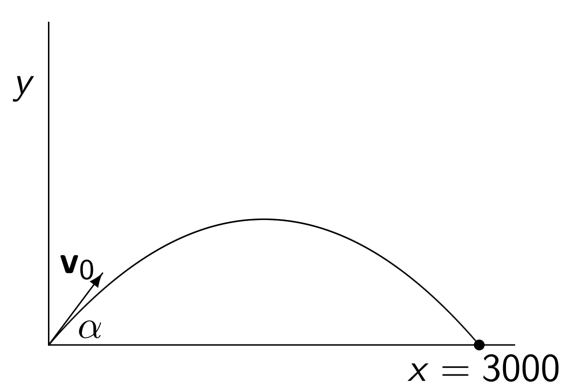


1. Let $\mathbf{a} = -g\mathbf{j}$, where g is the acceleration due to gravity.
2. Let the initial position be $\mathbf{r}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j}$.
3. Let the initial velocity be
$$\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}.$$
4. Then $\mathbf{v} = -gt\,\mathbf{j} + \mathbf{v}_0$ and
$$\overrightarrow{OP} = \mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} + \mathbf{v}_0\,t + \mathbf{r}_0.$$

Parametrically: $x = (v_0 \cos\alpha)t + x_0$; $y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0$.

Example 2: $x = (v_0 \cos\alpha)t$; $y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t$

Find the angle, α, at which a cannon must be aimed to hit a target 3 km down range, if the cannon's muzzle speed is 200 m/sec.

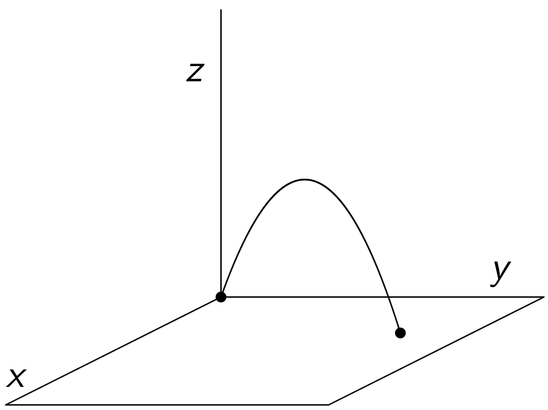


$x_0 = 0, y_0 = 0, v_0 = 200$

$$y = 0 \Rightarrow t = 0 \text{ or } t = \frac{2v_0 \sin\alpha}{g}$$
$$x = 3000 \Rightarrow 3000 = \frac{2v_0^2 \cos\alpha \sin\alpha}{g}$$
$$\Rightarrow \sin(2\alpha) = \frac{3000g}{v_0^2}$$
$$= 0.735$$
$$\Rightarrow 2\alpha \simeq 47.3^\circ \text{ or } 132.7^\circ$$
$$\Rightarrow \alpha \simeq 23.65^\circ \text{ or } 66.35^\circ$$

Example 3: Units in Feet

A ball is thrown into the air with initial velocity $\mathbf{v}_0 = 80\,\mathbf{j} + 80\,\mathbf{k}$. Due to the spin of the ball, its acceleration is $\mathbf{a} = 2\,\mathbf{i} - 32\,\mathbf{k}$. Determine where and with what speed the ball lands.



Take

$$g = 32 \text{ ft/sec}^2.$$

Ground level is taken to be

$$z = 0.$$

Distances are measured in feet.

Solution to Example 3

We have $\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{v}_0 = 80\,\mathbf{j} + 80\,\mathbf{k}$. Then

$$\begin{aligned} \mathbf{v} = \int \mathbf{a}\,dt &= \int (2\,\mathbf{i} - 32\,\mathbf{k})\,dt \\ &= 2t\,\mathbf{i} - 32t\,\mathbf{k} + \mathbf{v}_0 \\ &= 2t\,\mathbf{i} + 80\,\mathbf{j} + (80 - 32t)\,\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} \mathbf{r} = \int \mathbf{v}\,dt &= \int (2t\,\mathbf{i} + 80\,\mathbf{j} + (80 - 32t)\,\mathbf{k})\,dt \\ &= t^2\,\mathbf{i} + 80t\,\mathbf{j} + (80t - 16t^2)\,\mathbf{k}. \end{aligned}$$

$z = 0 \Rightarrow 80t - 16t^2 = 0 \Rightarrow t = 0$ or $t = 5$. At $t = 5 : x = 25$, $y = 400$, and $\mathbf{v} = 10\,\mathbf{i} + 80\,\mathbf{j} - 80\,\mathbf{k}$. So speed is $|\mathbf{v}| \simeq 113.6$.

Length of a Parametric Curve in $\mathbb{R}^2$; Alternate Derivation

If $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $\alpha < \beta$, the length of the curve is given by

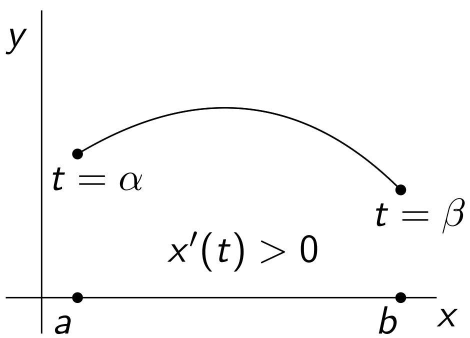
$$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

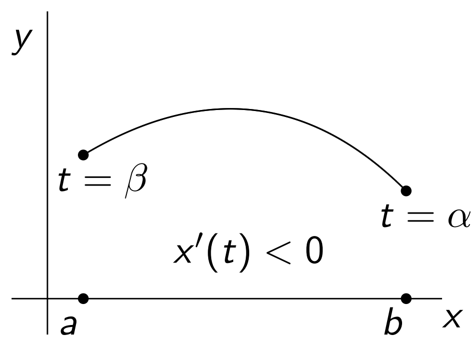
Proof: from Section 6.5,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{y'(t)}{x'(t)}\right)^2}\,dx = \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{(x'(t))^2}}\,dx$$

$$= \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{(x'(t))^2}}\,x'(t)\,dt = \sqrt{(x'(t))^2 + (y'(t))^2} \cdot \frac{x'(t)}{|x'(t)|}\,dt$$





x is going from left to right, and

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt,$$

$$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

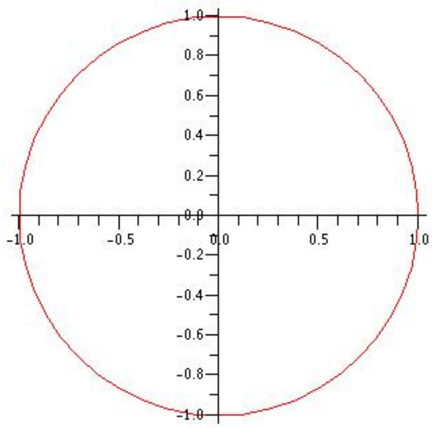
x is going from right to left, and

$$ds = -\sqrt{(x'(t))^2 + (y'(t))^2}\,dt,$$

$$L = -\int_\beta^\alpha \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

These are both the same, since $\int_\alpha^\beta f(t)\,dt = -\int_\beta^\alpha f(t)\,dt$.

Example 1: Circumference of Circle $x = a\cos t, y = a\sin t$



$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$$

$$\Rightarrow \quad ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$\begin{aligned} C &= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} a\, dt, \text{ if } a > 0 \\ &= 2\pi a \end{aligned}$$

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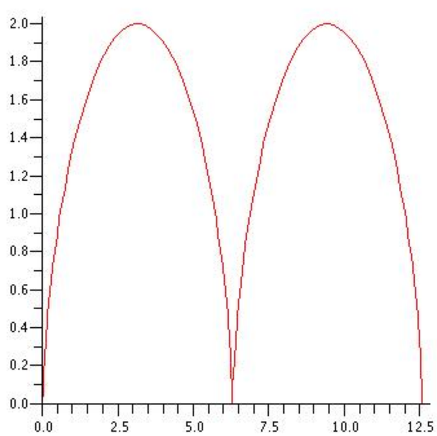
Example 2: Length of One Arch of a Cylcoid

$x = a(t - \sin t), y = a(1 - \cos t)$, for $0 \le t \le 2\pi$.



$$\begin{aligned} ds &= \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\, dt \\ &= a\sqrt{2 - 2\cos t}\, dt \\ &= a\sqrt{4\sin^2(t/2)}\, dt \\ &= 2a\sin(t/2)\, dt \end{aligned}$$

So

$$\begin{aligned} L &= \int_0^{2\pi} 2a\sin(t/2)\, dt \\ &= 4a\left[-\cos(t/2)\right]_0^{2\pi} = 8a \end{aligned}$$

Length of a Curve in $\mathbb{R}^3$

So if a parametric curve in $\mathbb{R}^2$ has position vector

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j},$$

then its length, for $\alpha < t < \beta$, is given by

$$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

This formula can easily be extended to a parametric curve in $\mathbb{R}^3$:

$$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

Vector Interpretation of the Length Formula

The tangent vector to the curve with position vector $\mathbf{r}$ is given by

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k},$$

and its length is given by

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Thus

$$L = \int_\alpha^\beta \left|\frac{d\mathbf{r}}{dt}\right|\,dt.$$

Example 3

Find the length of the helix

$$x = \cos(t), y = \sin(t), z = t$$

for $0 \leq t \leq 2\pi$. **Solution:**

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt \\ &= \int_0^{2\pi} \sqrt{2}\, dt \\ &= 2\pi\sqrt{2} \end{aligned}$$

Example 4

Find the length of the curve with parametric equations

$$x = 2e^t, y = e^{-t}, z = 2t, \text{ for } 0 \leq t \leq 1.$$

Solution:

$$\begin{aligned} L &= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \\ &= \int_0^1 \sqrt{\left(2e^t\right)^2 + \left(-e^{-t}\right)^2 + 2^2}\, dt \\ &= \int_0^1 \sqrt{4e^{2t} + e^{-2t} + 4}\, dt = \int_0^1 \sqrt{\left(2e^t + e^{-t}\right)^2}\, dt \\ &= \int_0^1 \left(2e^t + e^{-t}\right)\, dt = \left[2e^t - e^{-t}\right]_0^1 = 2e - e^{-1} - 1 \end{aligned}$$

Displacement and Distance Travelled

Suppose the position of a particle at time t for $a \leq t \leq b$ is given by the position vector $\mathbf{r}$. Then:

1. $\int_a^b \mathbf{v}\, dt = \mathbf{r}(b) - \mathbf{r}(a)$. That is, integrating the velocity of the particle over the time interval $[a, b]$ gives the displacement, or net change in position, of the particle.
2. $\int_a^b |\mathbf{v}|\, dt$ gives the total distance travelled by the particle along its trajectory over the time interval $[a, b]$. That is, since

$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|,$$

integrating the speed of the particle over the time inteval $[a, b]$ gives the total length of the particle's trajectory, which is the same as the total distance travelled.

Example 5

Suppose that the velocity at time t of a particle is given by

$$\mathbf{v} = -5\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} + 2\sqrt{6}\,\mathbf{k}$$

for $0 \leq t \leq 2\pi$. Then its net displacement is

$$\int_0^{2\pi} \mathbf{v}\, dt = \left[5\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 2\sqrt{6}t\,\mathbf{k}\right]_0^{2\pi} = 4\sqrt{6}\,\pi\,\mathbf{k}.$$

The total distance travelled is given by

$$\begin{aligned}\int_0^{2\pi} |\mathbf{v}|\, dt &= \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (5\cos t)^2 + (2\sqrt{6})^2}\, dt \\ &= \int_0^{2\pi} \sqrt{25 + 24}\, dt = [7t]_0^{2\pi} = 14\,\pi.\end{aligned}$$

Length of a Polar Curve

$x = r\cos\theta, y = r\sin\theta$ are the parametric equations of a polar curve. So

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

and

$$\begin{aligned} ds &= \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta \\ &= \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta, \text{ as you may check.} \end{aligned}$$

So the length of a polar curve is given by $\displaystyle\int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$

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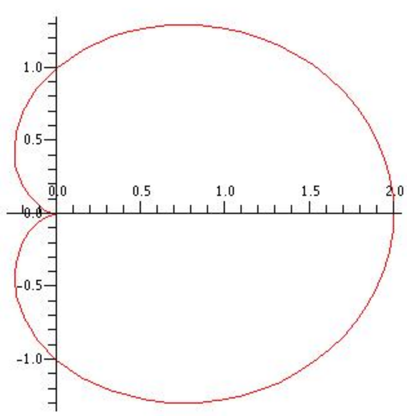
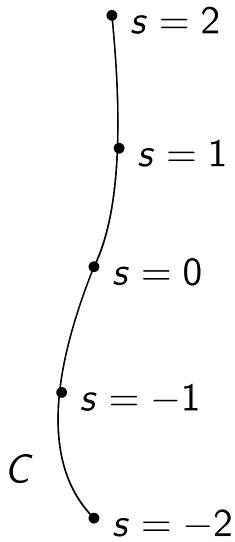
Example 6: Circumference of the Cardioid $r = 1 + \cos\theta$



$$\begin{aligned} C &= \int_{-\pi}^{\pi} \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\, d\theta \\ &= \int_{-\pi}^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= \int_{-\pi}^{\pi} \sqrt{4\cos^2(\theta/2)}\, d\theta \\ &= \int_{-\pi}^{\pi} 2\cos(\theta/2)\, d\theta \\ &= [4\sin(\theta/2)]_{-\pi}^{\pi} \\ &= 8 \end{aligned}$$

Note: $\int_0^{2\pi} 2\cos(\theta/2)\, d\theta = [4\sin(\theta/2)]_0^{2\pi} = 0$. Why is this wrong?

Arc Length as a Parameter



1. Pick an arbitrary reference point on the graph of C. This point will correspond to $s = 0$.
2. Starting from the reference point, pick one direction along the curve to be the positive direction.
3. If P is on the curve, let s be the 'signed' length along the curve from the reference point to P :
   $s > 0$ if P is in the positive direction from the reference point;
   $s < 0$ if P is in the negative direction from the reference point.

Example 7; Helix of Ex. 3 with Arc Length as Parameter

Let s be the length of the helix measured along the curve from the point $(1, 0, 0)$ to the point $(x, y, z) = (\cos(t), \sin(t), t)$, with positive direction chosen as up. Then

$$\begin{aligned} s &= \int_0^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\, du \\ &= \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2}\, du \\ &= \int_0^t \sqrt{2}\, du = t\sqrt{2} \end{aligned}$$

So $t = s/\sqrt{2}$, and in terms of its arc length s, the helix is given by

$$x = \cos(s/\sqrt{2}), y = \sin(s/\sqrt{2}), z = s/\sqrt{2}.$$

Change of Parameter

Suppose a curve in space is parametrized by two different parameters, t and τ. Then

$$\begin{aligned}
\frac{d\mathbf{r}}{d\tau} &= \left\langle \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right\rangle \\
&= \left\langle \frac{dx}{dt}\frac{dt}{d\tau}, \frac{dy}{dt}\frac{dt}{d\tau}, \frac{dz}{dt}\frac{dt}{d\tau} \right\rangle, \text{ by the chain rule} \\
&= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \frac{dt}{d\tau} \\
&= \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} \\
\Rightarrow \left|\frac{d\mathbf{r}}{d\tau}\right| &= \left|\frac{dt}{d\tau}\right| \left|\frac{d\mathbf{r}}{dt}\right|
\end{aligned}$$

Example 8

Consider the two parametrizations of the helix:

$$\mathbf{r} = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$$

and

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

Then

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2},$$

$$\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\right)^2 + \left(\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$

Finding Arc Length Parametrizations

Let C be the graph of a smooth curve in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) defined by the vector $\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then the formula

$$s = \int_{t_0}^{t} \left| \frac{d\mathbf{r}}{du} \right| du$$

defines a positive change of parameter from t to s, where s is the arc length parameter with $\mathbf{r}(t_0)$ as its reference point. That is,

$$\frac{ds}{dt} > 0.$$

Why is this derivative positive? By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right| > 0.$$

Example 9; Example 4 Revisited

Find a positive change of parameter from t to s, where s is the arc length parameter with $(2, 1, 0)$ as its reference point, for the curve with parametric equations

$$x = 2e^t, y = e^{-t}, z = 2t.$$

Solution:

$$\begin{aligned} s = \int_0^t \left| \frac{d\mathbf{r}}{du} \right| du &= \int_0^t \sqrt{(2e^u)^2 + (-e^{-u})^2 + 2^2}\, du \\ &= \int_0^t \sqrt{(2e^u + e^{-u})^2}\, du = \int_0^t \left(2e^u + e^{-u}\right) du \\ &= \left[2e^u - e^{-u}\right]_0^t = 2e^t - e^{-t} - 1 \end{aligned}$$

Arc Length Parametrization for Example 4

We can solve for t in terms of s by using the quadratic formula:

$$s = 2e^t - e^{-t} - 1 \Leftrightarrow se^t = 2e^{2t} - 1 - e^t$$

$$\Leftrightarrow 2e^{2t} - (1+s)e^t - 1 = 0 \Leftrightarrow e^t = \frac{1 + s + \sqrt{s^2 + 2s + 9}}{4}, \text{ since } e^t > 0.$$

Thus:

$$x = \frac{1 + s + \sqrt{s^2 + 2s + 9}}{2}, \; y = \frac{4}{1 + s + \sqrt{s^2 + 2s + 9}},$$

$$z = 2\ln\left(\frac{1 + s + \sqrt{s^2 + 2s + 9}}{4}\right).$$

This point is s units along the curve from the point $(2, 1, 0)$, for which $s = 0$. For example:

$$s = -7 \Rightarrow (x, y, z) = \left(-3 + \sqrt{11}, \frac{2}{-3 + \sqrt{11}}, -2\ln\left(\frac{-3 + \sqrt{11}}{2}\right)\right)$$

is 7 units before the point $(2, 1, 0)$;

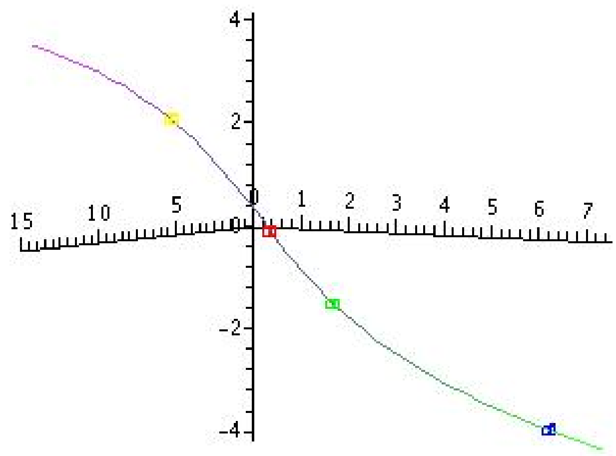
$$s = -2 \Rightarrow (x, y, z) = (1, 2, -2\ln 2)$$

is 2 units before the point $(2, 1, 0)$; and

$$s = 5 \Rightarrow (x, y, z) = \left(3 + \sqrt{11}, \frac{2}{3 + \sqrt{11}}, 2\ln\left(\frac{3 + \sqrt{11}}{2}\right)\right)$$

is 5 units after the point $(2, 1, 0)$ — as measured along the curve. See the graph on the next page.

Graph for Examples 4 and 9



Summary: Properties of Arc Length Parametrizations

Let C be the graph of a smooth vector-valued function $\mathbf{r}(t)$ in $\mathbb{R}^3$ or $\mathbb{R}^2$. Then

- $\left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt}$ if $s = \displaystyle\int_{t_0}^{t} \left|\dfrac{d\mathbf{r}}{du}\right| \, du$
- $\left|\dfrac{d\mathbf{r}}{ds}\right| = 1$
- If $\left|\dfrac{d\mathbf{r}}{dt}\right| = 1$, for all values of t, then for any value t_0

$$s = t - t_0$$

is an arc length parameter that has its reference point at a point on C where $t = t_0$.

Curvature

Let $\mathbf{r}(t) = \langle x, y, z\rangle$, where x, y, z are functions of t. Recall that the unit tangent vector $\mathbf{T}(t)$ is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|},$$

if we agree to consider the parameter t as time, and $\mathbf{v}(t) = \mathbf{r}'(t)$ as the velocity of a particle moving along the curve defined by the position vector $\mathbf{r}(t)$. The curvature at a point is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to the length of the curve s; it is symbolized by the Greek letter κ:

$$\kappa(s) = \left|\frac{d\,\mathbf{T}}{ds}\right|.$$

A Curvature Formula

Even though κ is defined in terms of the arc length s it is possible to find a formula for κ that does not require the curve to be parameterized in terms of its arc length. We use the chain rule:

$$\frac{d\,\mathbf{T}}{dt} = \frac{d\,\mathbf{T}}{ds}\cdot\frac{ds}{dt} \Rightarrow \kappa = \left|\frac{d\,\mathbf{T}}{ds}\right| = \frac{|d\,\mathbf{T}/dt|}{|ds/dt|} = \frac{1}{|\mathbf{v}|}\left|\frac{d\,\mathbf{T}}{dt}\right|,$$

since ds/dt, the rate of change of length along the curve with respect to time t, is the speed of the particle. Alternatively, one can also write κ as

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Example 1

Lines have zero curvature. **Proof:** let

$$\mathbf{r}(t) = \langle x_0 + a\,t, y_0 + b\,t, z_0 + c\,t\rangle,$$

for parameter t. Then $\mathbf{r}'(t) = \langle a, b, c\rangle$, so $|\mathbf{r}'(t)| = \sqrt{a^2 + b^2 + c^2}$, and

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle a, b, c\rangle}{\sqrt{a^2 + b^2 + c^2}}.$$

Since this is constant, with derivative $\mathbf{0}$, the curvature of the line is

$$\kappa = 0.$$

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Example 2

Circles have constant curvature. **Proof:** let

$$\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle,$$

for parameter $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = \langle -R\sin t, R\cos t\rangle$, so

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -R\sin t, R\cos t\rangle}{R} = \langle -\sin t, \cos t\rangle.$$

Thus

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\langle -\cos t, -\sin t\rangle|}{R} = \frac{1}{R};$$

the curvature of a circle is constant, it is the reciprocal of its radius.

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Alternative Curvature Formula

Let $\mathbf{r}(t)$ be the position of an object moving along a smooth curve; let $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ be its velocity and acceleration, respectively. Then

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}, \text{ or more briefly, } \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

Proof: $\mathbf{v} = |\mathbf{v}|\,\mathbf{T}$, by definition of $\mathbf{T}$. Differentiate both sides wrt t:

$$\mathbf{a} = \frac{d\,\mathbf{v}}{dt} = \frac{d(|\mathbf{v}|\,\mathbf{T})}{dt} = \mathbf{T}\frac{d|\mathbf{v}|}{dt} + |\mathbf{v}|\frac{d\,\mathbf{T}}{dt}.$$

Then

$$\mathbf{v} \times \mathbf{a} = |\mathbf{v}|\,\mathbf{T} \times \left(\mathbf{T}\frac{d|\mathbf{v}|}{dt} + |\mathbf{v}|\frac{d\,\mathbf{T}}{dt}\right) = \mathbf{0} + |\mathbf{v}|^2\,\mathbf{T} \times \frac{d\,\mathbf{T}}{dt}.$$

So

$$\begin{aligned} |\mathbf{v} \times \mathbf{a}| &= \left| |\mathbf{v}|^2\,\mathbf{T} \times \frac{d\,\mathbf{T}}{dt} \right| \\ &= |\mathbf{v}|^2\,|\mathbf{T}| \left| \frac{d\,\mathbf{T}}{dt} \right|, \text{ since}^1\ T \perp \frac{d\,\mathbf{T}}{dt} \\ &= |\mathbf{v}|^2 \left| \frac{d\,\mathbf{T}}{dt} \right|, \text{ since } |\mathbf{T}| = 1 \\ &= |\mathbf{v}|^2\,\kappa\,|\mathbf{v}|, \text{ using our first formula for } \kappa \end{aligned}$$

Thus

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

[1] by Example 2, Section 12.6 and the fact that $|\mathbf{T}| = 1$

Example 3: Curvature of a Helix

Consider the helix with vector equation $\mathbf{r}(t) = \langle a \cos t, a \sin t, b\, t \rangle$, for $a, b > 0$. Then

$$\mathbf{v} = \langle -a \sin t, a \cos t, b \rangle; \ \ \mathbf{a} = \langle -a \cos t, -a \sin t, 0 \rangle,$$

$$\mathbf{v} \times \mathbf{a} = \langle ab \sin t, -ab \cos t, a^2 \rangle,$$

and

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2 b^2 + a^4}}{(\sqrt{a^2 + b^2})^3} = \frac{a}{a^2 + b^2}.$$

Curvature of a Curve in the Plane

Parameterize the function $y = f(x)$ by $\mathbf{r} = \langle t, f(t), 0 \rangle$. Then the curvature of f is given by

$$\begin{aligned} \kappa &= \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \\ &= \frac{|\langle 1, f'(t), 0 \rangle \times \langle 0, f''(t), 0 \rangle|}{|\langle 1, f'(t), 0 \rangle|^3} \\ &= \frac{|\langle 0, 0, f''(t) \rangle|}{\left(\sqrt{1 + (f'(t))^2}\right)^3} \\ &= \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}} \end{aligned}$$

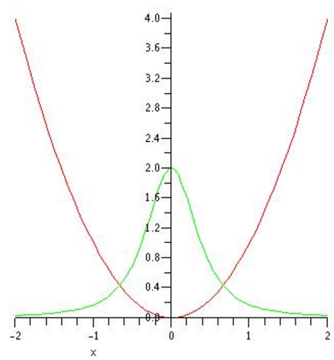
Example 4

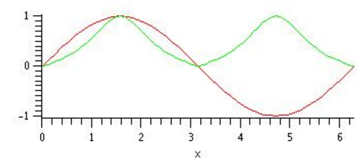
If $f(x) = x^2$, then

$$\kappa = \frac{2}{(1+4x^2)^{3/2}}.$$

If $f(x) = \sin x$, then

$$\kappa = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}.$$





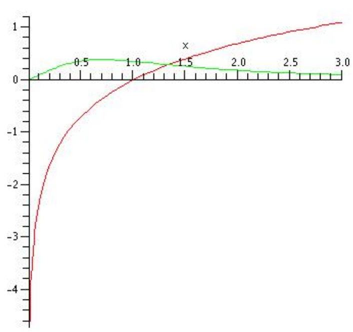
Example 5

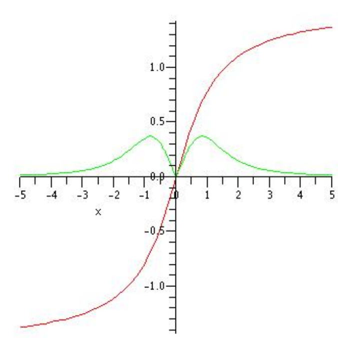
If $f(x) = \ln x$, then

$$\kappa = \frac{x}{(1+x^2)^{3/2}}.$$

If $f(x) = \arctan x$, then

$$\kappa = \frac{2|x|(1+x^2)}{(1+(1+x^2)^2)^{3/2}}.$$





Principal Unit Normal Vector

The curvature at a point on a curve tells you how *fast* a curve is turning. To describe the *direction* in which a curve is changing we use the principal unit normal vector, **N**, which is defined at a point P at which $\kappa \neq 0$, as follows:

$$\mathbf{N}(s) = \frac{d\,\mathbf{T}/ds}{|d\,\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

In terms of the parameter t, this can be written as

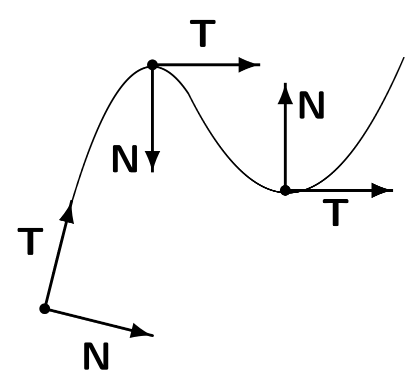
$$\mathbf{N}(t) = \frac{d\,\mathbf{T}/dt}{|d\,\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to the point P.

Properties of the Principal Unit Normal Vector

Let $\mathbf{r}(t)$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}(t)$ and principal unit normal vector $\mathbf{N}(t)$, Then

1. $|\mathbf{T}(t)| = 1$ and $|\mathbf{N}(t)| = 1$;
2. $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$, at all points where **N** is defined;
3. **N** points to the inside of the curve, ie. in the direction the curve is turning.



Example 6

Consider the helix with vector equation $\mathbf{r}(t) = \langle a\cos t, a\sin t, b\,t\rangle$, for $a, b > 0$. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -a\sin t, a\cos t, b\rangle}{|\langle -a\sin t, a\cos t, b\rangle|} = \frac{\langle -a\sin t, a\cos t, b\rangle}{\sqrt{a^2+b^2}}.$$

To calculate $\mathbf{N}(t)$, we first calculate

$$\frac{d\,\mathbf{T}(t)}{dt} = \frac{\langle -a\cos t, -a\sin t, 0\rangle}{\sqrt{a^2+b^2}},$$

and then

$$\left|\frac{d\,\mathbf{T}(t)}{dt}\right| = \frac{\sqrt{a^2+0^2}}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}.$$

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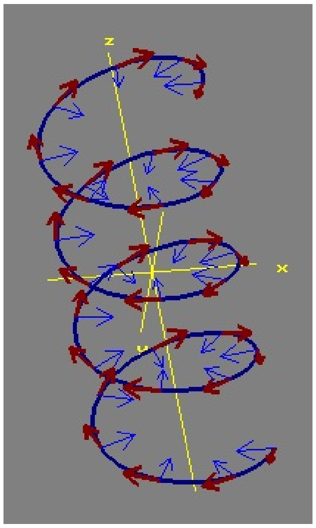
Thus the principal unit normal for a helix is

$$\mathbf{N}(t) = \frac{d\,\mathbf{T}/dt}{|d\,\mathbf{T}/dt|} = \langle -\cos t, -\sin t, 0\rangle.$$

The graph to the right shows a helix with unit tangent vectors in red and principal unit normal vectors in blue.



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Components of Acceleration in Terms of N and T

If $\mathbf{a}$ is the acceleration of a particle moving along a curve with position vector $\mathbf{r}$, at time t, then it turns out that $\mathbf{a}$ must be a linear combination of $\mathbf{T}$ and $\mathbf{N}$. That is, the vector $\mathbf{a}$ must be in the plane spanned by the two vectors $\mathbf{N}$ and $\mathbf{T}$. Why? Firstly,

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{v} = \mathbf{T}\,|\mathbf{v}| = \mathbf{T}\,\frac{ds}{dt}.$$

Secondly, by the product rule and the chain rule,

$$\mathbf{a} = \frac{d\,\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\,\frac{ds}{dt}\right) = \frac{d\,\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\,\frac{d^2s}{dt^2} = \frac{d\,\mathbf{T}}{ds}\frac{ds}{dt}\frac{ds}{dt} + \mathbf{T}\,\frac{d^2s}{dt^2}.$$

Finally, use $|\mathbf{v}| = ds/dt$ and $\kappa\,\mathbf{N} = d\mathbf{T}/ds$ to rewrite $\mathbf{a}$ as

$$\mathbf{a} = \kappa\,|\mathbf{v}|^2\,\mathbf{N} + \frac{d^2s}{dt^2}\,\mathbf{T}.$$

This establishes that $\mathbf{a}$ is indeed in the plane spanned by $\mathbf{N}$ and $\mathbf{T}$. We write

$$\mathbf{a} = a_N\,\mathbf{N} + a_T\,\mathbf{T},$$

where

- $a_T = \dfrac{d^2s}{dt^2}$ is called the tangential component of the acceleration; it measures the particle's change in speed.
- $a_N = \kappa\,|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ is called the normal component of the acceleration; it measures the particle's change in direction.

Example 7: Circular Motion with Constant Speed

Let $\mathbf{r} = \langle R\cos(\omega t), R\sin(\omega t)\rangle$, for positive constants R, ω. Then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -\omega R\sin(\omega t), \omega R\cos(\omega t)\rangle; \ \frac{ds}{dt} = |\mathbf{v}(t)| = R\omega.$$

By Example 2 the curvature of a circle is $\kappa = 1/R$. So

$$a_T = \frac{d^2s}{dt^2} = 0; \ a_N = \kappa\,|\mathbf{v}|^2 = \frac{1}{R}(R\omega)^2 = R\omega^2.$$

Thus for circular motion with constant speed, the acceleration is entirely in the normal direction, orthogonal to the tangent vectors.

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Example 8: Parabolic Motion

If $\mathbf{r}(t) = \langle t, t^2\rangle$, then $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2\rangle$ and

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}; \ \mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{\langle -2t, 1\rangle}{\sqrt{1+4t^2}},$$

as you may check. Since $\mathbf{N}, \mathbf{T}$ are othonormal we can find a_N and a_T directly, as in MAT188:

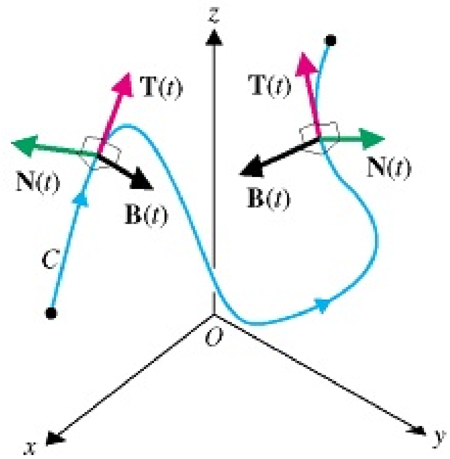
$$a_N = \mathbf{a}\cdot\mathbf{N} = \frac{2}{\sqrt{1+4t^2}}; \ a_T = \mathbf{a}\cdot\mathbf{T} = \frac{4t}{\sqrt{1+4t^2}}.$$

Note that at the origin, where curvature of the parabola is greatest (see Example 4), the normal component of the acceleration is greatest while the tangential component of the acceleration is zero.

The Unit Binormal Vector B and the TNB Frame

Suppose $\mathbf{r}(t) = \langle x, y, z\rangle$ is a vector-valued function in $\mathbb{R}^3$. The vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ defines a third unit vector orthogonal to both $\mathbf{N}$ and $\mathbf{T}$. $\mathbf{B}$ is called the unit binormal vector.

The three vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ define an orthonormal basis, or frame, at each point on the curve determined by the parameter t. The three vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a right-handed coordinate system that changes its orientation as we move along the curve; it is called the **TNB** frame.



Properties of the Unit Binormal Vector B

- $d\mathbf{B}/ds$ is orthogonal to both $\mathbf{T}$ and $d\,\mathbf{N}/ds$:
$$\frac{d\,\mathbf{B}}{ds} = \frac{d\,(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\,\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\,\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\,\mathbf{N}}{ds}$$
- $d\,\mathbf{B}/ds$ is orthogonal to both $\mathbf{B}$ and $\mathbf{T}$:
$$|\mathbf{B}|^2 = 1 \Rightarrow 2\,\mathbf{B} \cdot \frac{d\,\mathbf{B}}{ds} = 0$$
- Therefore $d\,\mathbf{B}/ds$ is parallel to $\mathbf{N}$. We define the torsion of the curve at a point to be the scalar τ such that
$$\frac{d\,\mathbf{B}}{ds} = -\tau\,\mathbf{N} \Leftrightarrow \tau = -\frac{d\,\mathbf{B}}{ds} \cdot \mathbf{N}.$$

The Osculating Plane and Torsion

The plane spanned by the two vectors **T** and **N** is called the osculating plane. The rate at which the curve determined by **r** twists in or out of the osculating plane is measured by $d\,\mathbf{B}/ds$, the rate at which **B** changes with respect to the curve's length. From the previous slide we know that

$$\frac{d\,\mathbf{B}}{ds} = -\tau\,\mathbf{N}.$$

Since **N** is a unit vector,

$$\left|\frac{d\,\mathbf{B}}{ds}\right| = |-\tau|\,|\mathbf{N}| = |\tau|.$$

Thus the magnitude of the torsion is the magnitude of the rate at which the curve twists in or out of the osculating plane.

Example 9: Unit Binormal and Torsion of a Helix

Let $\mathbf{r}(t) = \langle a\cos t, a\sin t, b\,t\rangle$, for $a, b > 0$. From Example 6 we know that

$$\mathbf{T}(t) = \frac{\langle -a\sin t, a\cos t, b\rangle}{\sqrt{a^2+b^2}},\ \ \mathbf{N}(t) = \langle -\cos t, -\sin t, 0\rangle.$$

Then

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{\langle b\sin t, -b\cos t, a\rangle}{\sqrt{a^2+b^2}},$$

$$\frac{d\,\mathbf{B}}{ds} = \frac{d\,\mathbf{B}/dt}{ds/dt} = \frac{1}{|\mathbf{v}|}\frac{d\,\mathbf{B}}{dt} = \frac{\langle b\cos t, b\sin t, 0\rangle}{a^2+b^2},$$

and

$$\tau = -\frac{d\,\mathbf{B}}{ds}\cdot\mathbf{N} = \frac{b}{a^2+b^2}.$$

Summary: Formulas For Curves in Space

If $\mathbf{r}(t)$ is the position vector along a curve in terms of a parameter t, and s is the arc length measured along the curve, then

- the velocity is $\mathbf{v} = \mathbf{r}'(t)$.
- the speed is $|\mathbf{v}| = |\mathbf{r}'(t)| = ds/dt$.
- the acceleration is $\mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t)$.
- the unit tangent vector is $\mathbf{T} = \dfrac{d\,\mathbf{r}}{ds} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$.
- the principal unit normal vector is $\mathbf{N} = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.
- the unit binormal vector is $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \dfrac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$.

Chapter 12: Vector-Valued Functions
12.3 and 12.4 Review: Vectors in $\mathbb{R}^3$
12.5 Introduction to Vector-Valued Functions
12.6 Calculus of Vector-Valued Functions
12.7 Motion Along a Curve
12.8 Length of Curves
12.9 Curvature and Normal Vectors

- the curvature is $\kappa = \left|\dfrac{d\,\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\,\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$.
- the torsion is $\tau = -\dfrac{d\,\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$.
- the tangential and normal components of acceleration are a_T and a_N, respectively, with $\mathbf{a} = a_N\,\mathbf{N} + a_T\,\mathbf{T}$, and

$$a_N = \kappa\,|\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}, \quad a_T = \frac{d^2 s}{dt^2} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}.$$

Note we did not prove all of the above formulas! Some are left as exercises for the student.

Chapter 12 Lecture Notes MAT187H1F Lec0101 Burbulla