MAT186H1F LEC0103 Burbulla

Week 0 Lecture Notes

Fall 2021

General Advice

Introduction to Calculus

Written Tests

All tests and exams require you to show your work and explain what you are doing. The final answer, even if it is correct, will be worth very little unless you have *explained* your solution.

Moreover, final answers should always be simplified. Partial credit can be obtained for partially correct work, but NO credit may be given if your work

- ▶ is poorly presented,
- ▶ difficult to decipher,
- ▶ uses incorrect mathematical notation,
- ▶ or makes no sense.

It is not our responsibility to figure out what you have done; you are supposed to make it clear what you are doing. You can find posted model solutions to many old term tests and exams at

https://www.math.utoronto.ca/burbulla/detailed

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Homework

Going over the text book, viewing videos posted on line, and attending lectures, in whatever form they are, will expose you to new ideas and show you what kind of examples you should be able to do. But none of the material from lectures will really sink in until you try the homework.

> *To be successful in a math course you must do the homework.*

You can get help with homework from your lecturer during his or her office hours. Teaching Assistants may also be available for help, at times to be posted, either on-line or in a math aid drop-in centre. It is advisable to do your homework on a regular (weekly) basis and to attend tutorials. Leaving things until the night before a quiz or a test is asking for trouble.

Notation

Mathematics has its own set of symbols. If you use them, you must use them correctly. (You will lose marks on tests or exams if your notation is incorrect.) Here are some examples of common errors:

Example 1: suppose you are differentiating the function $y = x^2$. If you write

$$\begin{aligned} y &= x^2 \\ &= 2x \end{aligned}$$

it is incorrect, even though most math people would know what you are doing. Of course, you should write

$$\begin{aligned} y &= x^2 \\ \Rightarrow y' &= 2x \end{aligned}$$

Or you could simply write $\dfrac{dx^2}{dx} = 2x$.

Example 2: don't confuse equality with approximation:

$$\sqrt{2} \neq 1.414; \sqrt{2} \approx 1.414$$

Example 3: don't confuse implication ($\Rightarrow$) with equality ($=$). The symbol $\Rightarrow$ is a logical connective and means   If ... then ... For example,

$$\text{if } y = x^2 \text{ then } y' = 2x,$$

can be written as

$$y = x^2 \Rightarrow y' = 2x.$$

But writing something like

$$\begin{aligned} ax^2 + bx + c &\Rightarrow 0 \\ x &\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

is abuse of notation.

Example 4: the following expressions

$$\frac{0}{0}, \frac{\infty}{\infty}, 0^0, 0 \cdot \infty, 1^\infty, \infty^0 \text{ and } \infty - \infty$$

are indeterminate. When evaluating limits of this type, which may or may not exist, don't *equate* the limit to one of the above expressions. For example, writing

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{0}{0}$$

is incorrect; the limit is actually equal to $\frac{3}{2}$. You *can* say the limit is of the form $\frac{0}{0}$, or that when $x = 1$,

$$\frac{x^3 - 1}{x^2 - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0},$$

which is indeterminate.

Example 5: when reducing a matrix, don't put equal ($=$) signs between matrices which *aren't* equal:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

is incorrect. Instead, use an arrow ($\to$) or a tilde ($\sim$) between row equivalent matrices:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}.$$

Example 6: an indefinite integral always includes an arbitrary constant. To write

$$\int x^2 \, dx = \frac{x^3}{3}$$

is incomplete, and will cost you a mark every time. Correct is:

$$\int x^2 \, dx = \frac{x^3}{3} + C.$$

Example 7: the correct indefinite integral of $\dfrac{1}{x}$ with respect to x is:

$$\int \frac{1}{x} \, dx = \ln |x| + C.$$

Without the absolute value it is wrong, and will cost you a mark every time. Some students even write

$$\int \frac{1}{x} \, dx = \ln x,$$

which is doubly wrong – and will cost you *two* marks.

Logic and Proof

Errors in logic that show up in your written solutions will cost you marks on a test or exam, and may make your whole "solution" worthless – even if your calculations are correct.

- ▶ It seems as if many first year students have never been exposed to any formal logic in their high school curriculum.
- ▶ Unfortunately, logic is part and parcel of mathematics.
- ▶ You can't present a mathematical argument without using logic.
- ▶ You can't understand what a theorem says if you can't distinguish between hypotheses and conclusions.
- ▶ What follows is some logical terminology you should become familiar with.
- ▶ Let P and Q be statements, mathematical or otherwise.

Implication

$P \Rightarrow Q$ means "P implies Q," that is "If P is true, then Q is true." Most theorems in mathematics are of this form. Consider this version of the Intermediate Value Theorem:

> *If f is a continuous function on the closed interval $[a, b]$ such that $f(a)f(b) < 0$ then there is a number $c \in (a, b)$ such that $f(c) = 0$.*

The theorem is in the form $P \Rightarrow Q$, where P is the statement

> *f is a continuous function on the closed interval $[a, b]$ such that $f(a)f(b) < 0$*

This is the *hypothesis* of the theorem. Q is the statement

> *there is a number $c \in (a, b)$ such that $f(c) = 0$*

This is called the *conclusion* of the theorem.

Warning: there are many different ways in English to say "$P \Rightarrow Q$." Here are some of them:

1. Q is true if P is true.
2. For Q to be true it is sufficient for P to be true.
3. P is true only if Q is true.
4. For P to be true it is necessary that Q is true.

Negation: $\neg P$ represents the negation of P. The Law of the Excluded Middle states that every statement is true or its negation is true: one of P or $\neg P$ is always true. An application of this is proof by contradiction; that is, to show the statement A is true, assume that A is false and deduce a contradiction.

Equivalent Statements and Converse

Logical Equivalence: $P \Leftrightarrow Q$ means P and Q are equivalent statements. That is

"P is true if and only if Q is true."

Many important theorems in mathematics are of this form. For example,

Let A be an $n \times n$ matrix. A is invertible if and only if $\det(A) \neq 0$.

P is the statement "A is invertible;" Q is the statement "$\det(A) \neq 0$." The theorem means that these two statements are equivalent; for given A, they are either both true or both false.

Converse: the converse of the implication $P \Rightarrow Q$ is another implication, namely $Q \Rightarrow P$. Thus P and Q are equivalent if both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true.

Contrapositive

Contrapositive: the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. It is logically true that

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P);$$

i.e. an implication and its contrapositive are logically equivalent.
Example 8: The converse of a true statement is *not* necessarily a true statement. For example, it is a true statement that

If $x = 2$ then $x^2 = 4$.

The converse of this statement,

If $x^2 = 4$ then $x = 2$.

is false, since $x = -2$ is also possible. Of course, the contrapositive of the original statement—If $x^2 \neq 4$ then $x \neq 2$.—is also a true statement.

Example 9

To show a statement is *false* you only have to exhibit one
counterexample. For example, the statement

> *If the number of equations in a system of linear equations
> is less than the number of unknowns in the system, then
> the system has infinitely many solutions.*

is a false statement. Consider the example

$$
\begin{aligned}
x + y + z &= 0 \\
x + y + z &= 1
\end{aligned}
$$

which consists of two equations in three unknowns $(2 < 3)$ but has
no solutions, because it is inconsistent.

Example 10

On the other hand, to show a statement is *true,* it is not enough to
illustrate it with one example. Consider the statement

> *If A is a $n \times n$ matrix such that $A^2 = I$ then $\det(A) = \pm 1$.*

which *is* a true statement. You can't prove it is true by simply
considering one choice of A. Instead, you must give a general
argument:

$$
\begin{aligned}
A^2 = I \quad &\Rightarrow \quad \det\left(A^2\right) = \det(I) \\
&\Rightarrow \quad (\det(A))^2 = 1 \\
&\Rightarrow \quad \det(A) = \pm 1
\end{aligned}
$$

Inexcusable Algebraic Errors

Should any of the following types of algebraic errors show up in one of your solutions, everything after the appearance of the error will be forfeited. In computer jargon, these errors are "fatal errors."
Example 11: both $\sqrt{a^2} = a$ and $\sqrt{a^2} = \pm a$ are wrong! Correct is:

$$\sqrt{a^2} = |a|.$$

Example 12: $\sqrt{a^2 + b^2} = a + b$ is wrong. There just isn't any easy way to simplify $\sqrt{a^2 + b^2}$.
Example 13: $\dfrac{1}{a + b} = \dfrac{1}{a} + \dfrac{1}{b}$ is wrong. Correct for $a \neq 0, b \neq 0$ is

$$\frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab}.$$

Aside: for $a + b \neq 0$, $\dfrac{1}{a + b} = \dfrac{1}{a} + \dfrac{1}{b} \Leftrightarrow a^2 + ab + b^2 = 0$, which is *impossible* if a and b are non-zero real numbers.

Example 14: $\ln\left(\dfrac{M}{N}\right) = \dfrac{\ln M}{\ln N}$ and $\ln(MN) = \ln N \ln N$ are both wrong. Correct are

$$\ln\left(\frac{M}{N}\right) = \ln M - \ln N \text{ and } \ln(MN) = \ln M + \ln N,$$

if both $M > 0$ and $N > 0$.
Example 15: $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha + \cos\beta$ are both wrong. Correct are:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha,$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Example 16: $\sin(k\theta) = k\sin\theta$ is wrong, unless $k = -1, 0, 1$. There is no easy way to simplify $\sin(k\theta)$, but here are two special cases

$$\sin(2\theta) = 2\sin\theta\cos\theta \text{ and } \sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

Example 17: $\ln(M + N) = \ln M + \ln N$ is wrong. There just isn't any easy way to simplify $\ln(M + N)$.

Absolute Value

Probably the most misunderstood formula from high school is the definition of absolute value:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

That is, to simplify $|a|$ you must take cases.

Example 18: Thus $|x|^2 = x^2$ and $\sqrt{x^2} = |x|$ are correct, but $|x|^3 = x^3$ is incorrect. Can you explain why?

Some Comments About Marking: if you make a copying error – or some other 'dumb' mistake – in your work which results in the rest of the problem being much easier than intended, then you will forfeit lots of marks. On the other hand, if you make a copying error and the rest of the problem remains of comparable difficulty, then you will only lose a few marks.

Example 19

Here is an example: consider the indefinite integral

$$\int \frac{1}{x^3 + 1} \, dx,$$

which is a fairly involved problem requiring partial fractions, completing the square and a trig substitution to solve. Say it is worth 10 marks on a test. If you miscopy the integral as

$$\int \frac{1}{x^3 - 1} \, dx$$

you could still get 9 out of 10, if the rest of your work is correct, because the exact same procedures as the intended question are involved. However, if you miscopy the question as

$$\int \frac{1}{x^2 + 1} \, dx$$

it becomes a one-liner, and you would get at most 1 mark out of 10

Some Comments About Your Calculator

Don't blame your calculator if it gives you the wrong answer. For example, many calculators are programmed to accept only positive arguments when taking roots of a number. So, even though there is a real value for

$$(-1)^{(1/3)},$$

some calculators will not evaluate this expression, or will display an answer of 0. The correct answer is of course

$$(-1)^{(1/3)} = -1.$$

Make sure you know how to use your calculator, and its built-in limits.

The same holds true for WolframAlpha and other mathematical software, like *Matlab, Maple* or *Mathematica.* They aren't always programmed as you might expect.

High School Math We Assume You Know

Note: probably most of these formulas will be reviewed in your MAT186H1F lectures, time permitting. However, you are expected to understand them and be able to use them, when necessary, even if they aren't covered in your lectures.

Quadratic Formula: suppose a, b, c are real numbers. Then

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots are

- real and distinct, if $b^2 - 4ac > 0$;
- real and repeated, if $b^2 - 4ac = 0$;
- complex conjugates, if $b^2 - 4ac < 0$.

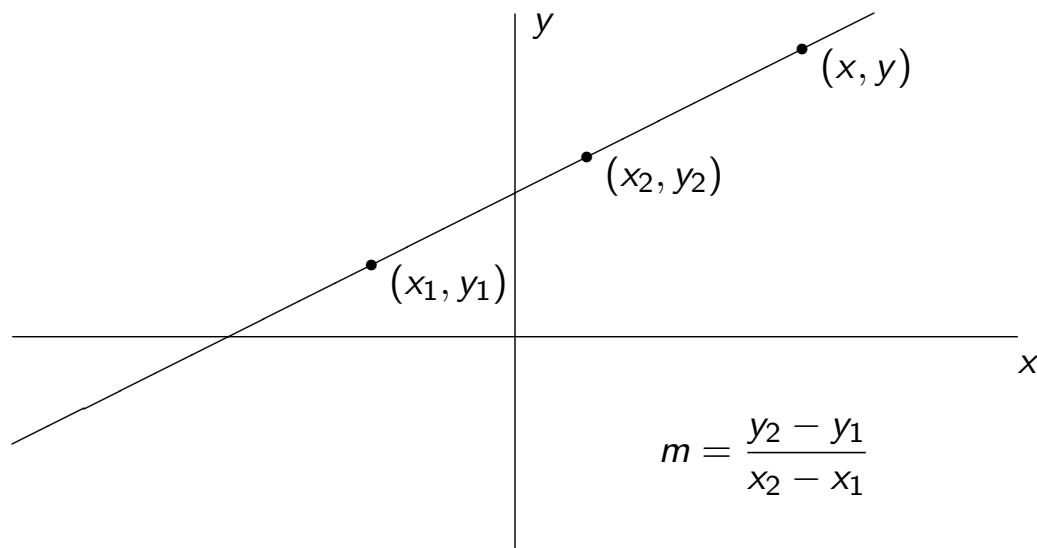## Slope, y-intercept Form of a Line

Slope, y-intercept form: $y = mx + b$

The graph shows a line crossing the y-axis at $(0, b)$ and passing through a point (x, y), with Δy and Δx marked, and

$$m = \frac{\Delta y}{\Delta x}$$
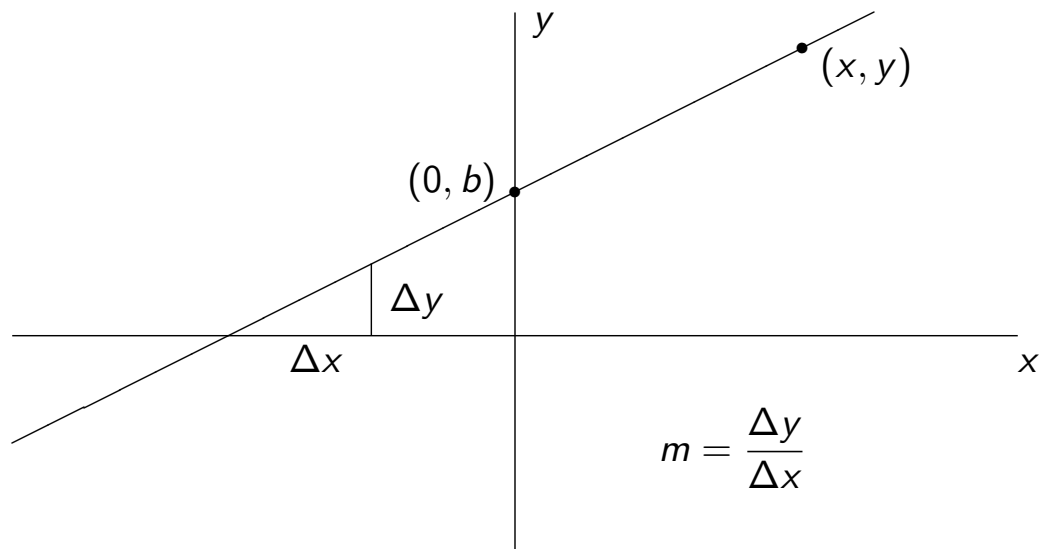
Slope-Point Form of a Line

$$\frac{y - y_1}{x - x_1} = m$$

The graph shows a line through points (x_1, y_1) and (x, y), with Δy and Δx marked, and

$$m = \frac{\Delta y}{\Delta x}$$
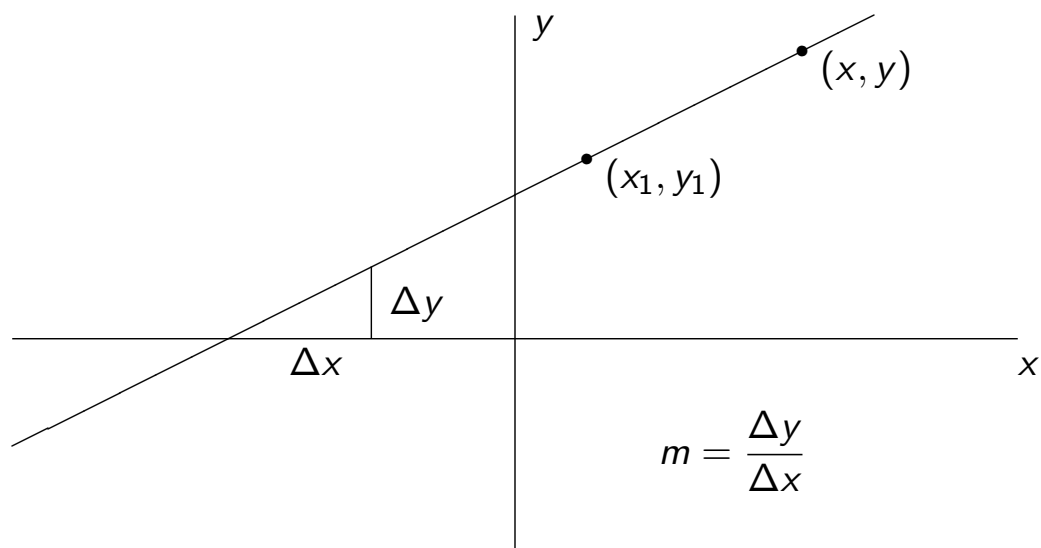
Two Points Form of a Line

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Intercept Form of a Line: for $ab \neq 0$.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$m = -\frac{b}{a}$$
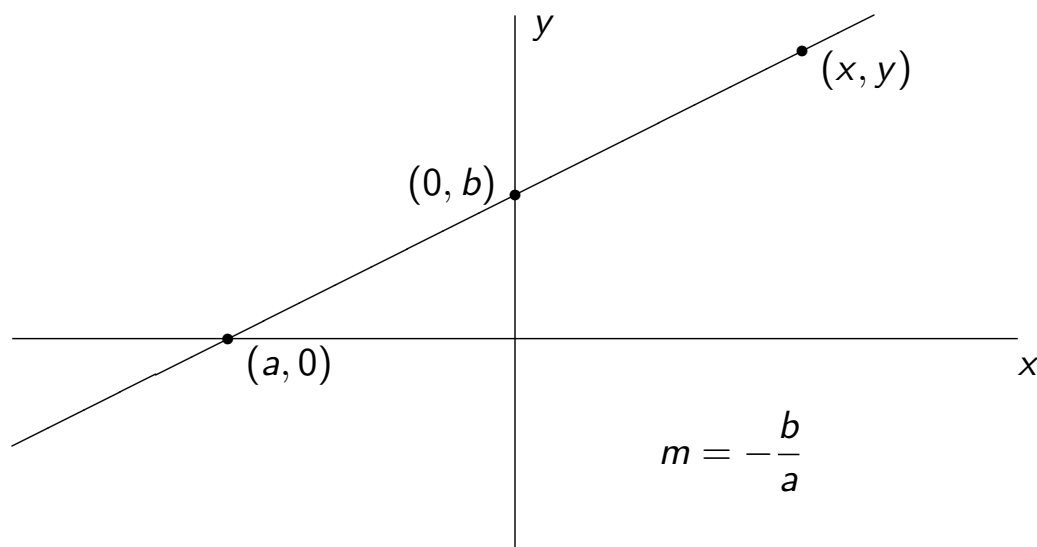
The Six Trigonometric Functions and Right Triangles
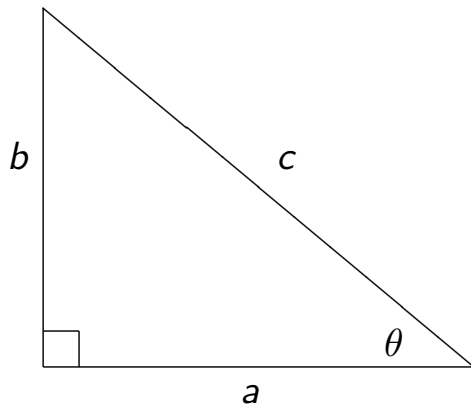
- $\sin\theta = \dfrac{b}{c}$
- $\cos\theta = \dfrac{a}{c}$
- $\tan\theta = \dfrac{b}{a} = \dfrac{\sin\theta}{\cos\theta}$
- $\csc\theta = \dfrac{c}{b} = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{c}{a} = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta}$

The Pythagorean Identities:

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

30-60-90 Triangle
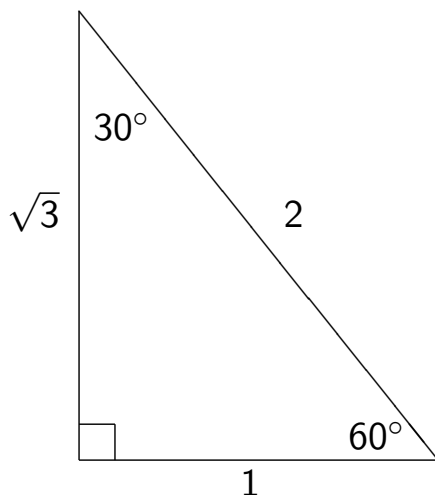
- $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$
- $\cos 60^\circ = \dfrac{1}{2}$
- $\tan 60^\circ = \sqrt{3}$
- $\sin 30^\circ = \dfrac{1}{2}$
- $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$
- $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$

45-45-90 Triangle
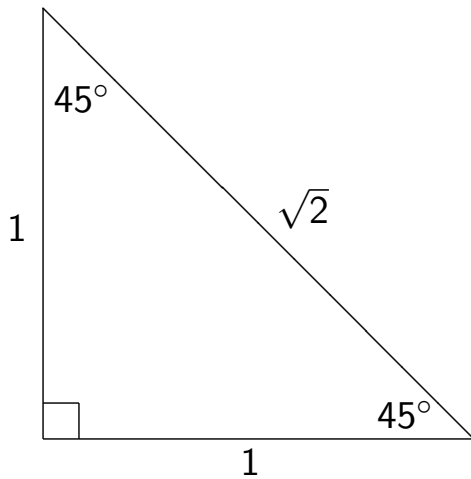
- $\sin 45° = \dfrac{1}{\sqrt{2}}$
- $\cos 45° = \dfrac{1}{\sqrt{2}}$
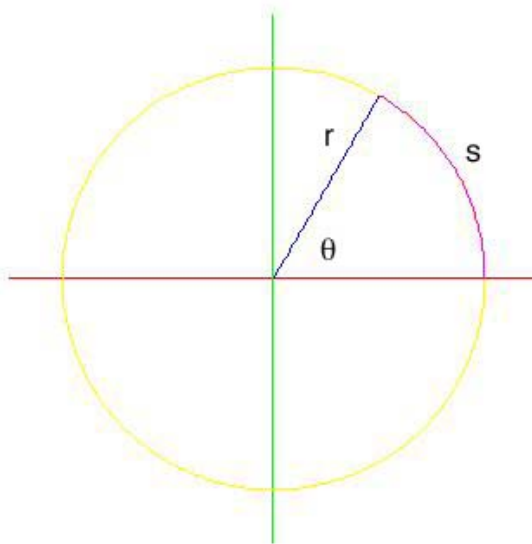- $\tan 45° = 1$

Sine Law and Cosine Law
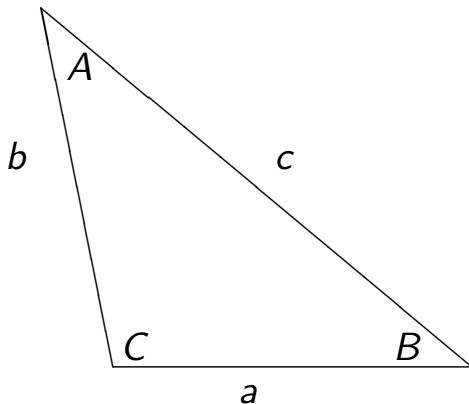
Sine Law:
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Law:
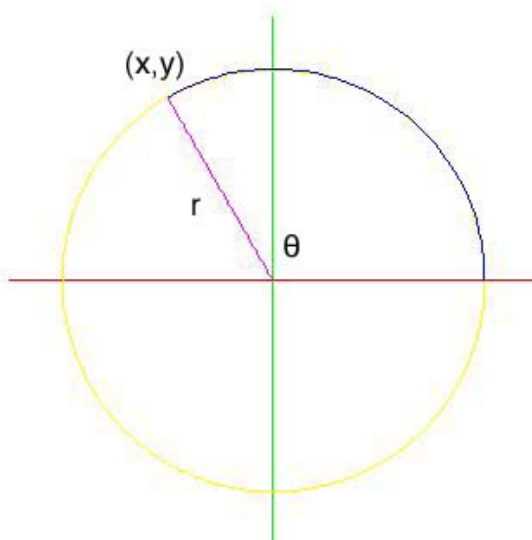- $a^2 = b^2 + c^2 - 2bc \cos A$
- $b^2 = a^2 + c^2 - 2ac \cos B$
- $c^2 = a^2 + b^2 - 2ab \cos C$

Radian Measure and Circles

- $\theta = \dfrac{s}{r}$
- $s = r\theta$
- π radians $= 180°$
- Radian measure is the ratio of two lengths; so it is unit free.

The Six Trigonometric Functions; θ in radians.

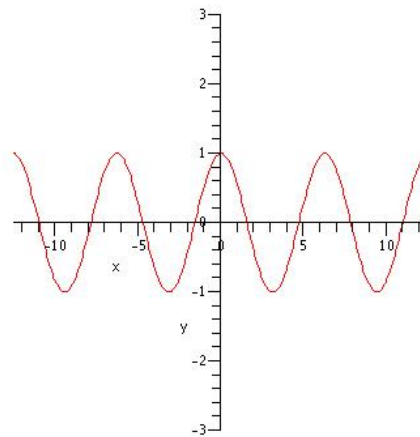- $\sin\theta = \dfrac{y}{r}$
- $\cos\theta = \dfrac{x}{r}$
- $\tan\theta = \dfrac{y}{x}$
- $\csc\theta = \dfrac{r}{y}$
- $\sec\theta = \dfrac{r}{x}$
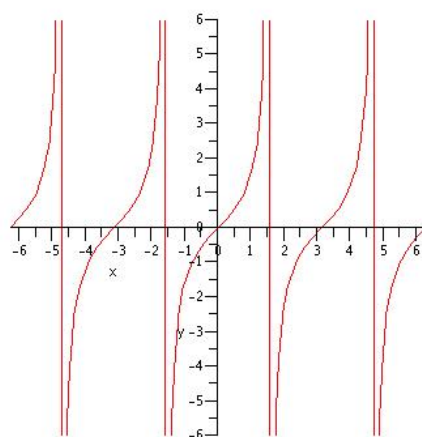- $\cot\theta = \dfrac{x}{y}$

Graphs of Sine and Cosine
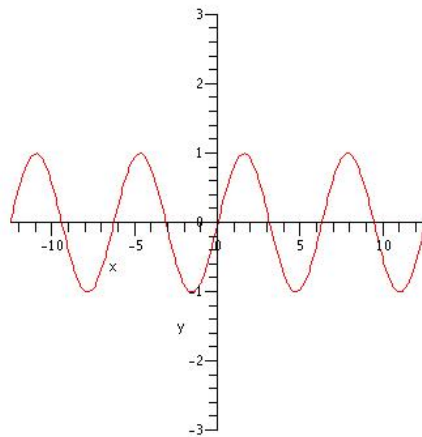
Graph of $y = \sin x$.
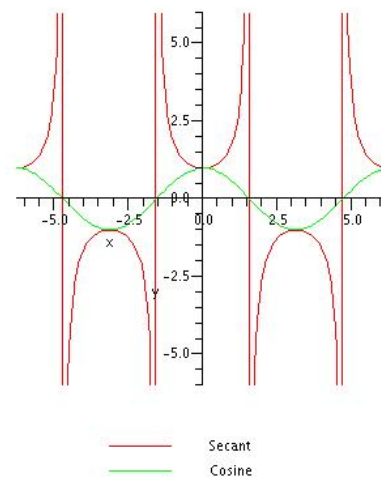**Domain:** $\mathbb{R}$; **Range:** $[-1, 1]$

Graph of $y = \cos x$.
Domain: $\mathbb{R}$; **Range:** $[-1, 1]$

Graph of Tangent and Secant

Graph of $y = \tan x$.
Domain: $x \neq \dfrac{(2k+1)}{2}\pi$;
Range: $\mathbb{R}$

Secant
Cosine

Graph of $y = \sec x$.
Domain: $x \neq \dfrac{(2k+1)}{2}\pi$;
Range: $|y| \geq 1$

Trigonometric Formulas

Complementary angles:

- $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$
- $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

Supplementary angles:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$

Negative angles:

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$

Sum and Difference Formulas:

- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
- $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Double Angle Formulas

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\cos(2\theta) = 1 - 2\sin^2\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$

Reformulation of Double Angle Formulas:

$$\bullet\,\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \qquad \bullet\,\cos^2\theta = \frac{1 + \cos(2\theta)}{2};$$

$$\bullet\,\sin^2\frac{A}{2} = \frac{1 - \cos A}{2} \qquad \bullet\,\cos^2\frac{A}{2} = \frac{1 + \cos A}{2}.$$
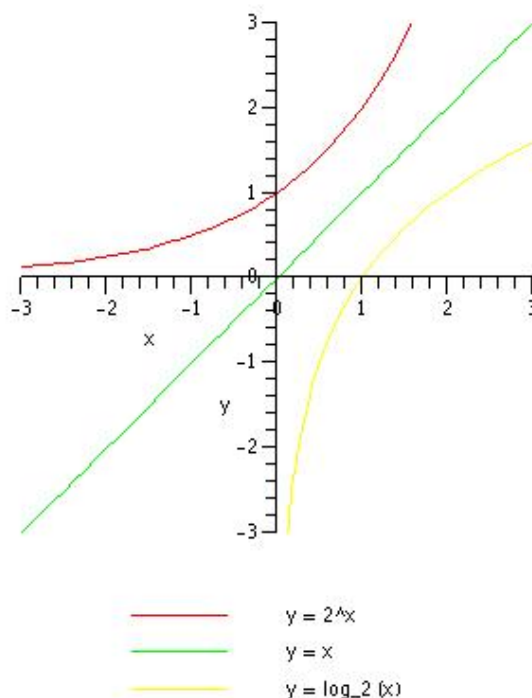
Logs and Exponentials

Properties of Exponentials: Let a, b be positive numbers; let x, y be real numbers. Then

- $a^{x+y} = a^x a^y$ •$a^{x-y} = \dfrac{a^x}{a^y}$ •$a^{xy} = (a^x)^y$ •$(ab)^x = a^x b^x$

- $a^0 = 1$ • $a^1 = a$ • $a^{-1} = \dfrac{1}{a}$.

Properties of Logarithms, for $a > 0, a \neq 1$: Let x and y be positive real numbers; let z be any real number. Then:

- $\log_a(xy) = \log_a x + \log_a y$ • $\log_a 1 = 0$

- $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$ • $\log_a a = 1$

- $\log_a(x^z) = z \log_a x$ • $\log_a\left(\dfrac{1}{a}\right) = -1$

- $\log_a a^z = z$ • $a^{\log_a x} = x$

Grahps of $y = 2^x$ and $y = \log_2 x$

—— $y = 2^x$
—— $y = x$
—— $y = \log_2(x)$

Calculus Formulas using Leibnizian Notation

Derivatives:

- $\dfrac{dc}{dx} = 0$

- $\dfrac{dx^n}{dx} = nx^{n-1}$

- $\dfrac{d\sin(x)}{dx} = \cos x$

- $\dfrac{d\cos(x)}{dx} = -\sin x$

- $\dfrac{de^x}{dx} = e^x$

- $\dfrac{d\ln(x)}{dx} = \dfrac{1}{x}$

Differentiation Rules:

- $\dfrac{d(f(x) + g(x))}{dx} = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$

- $\dfrac{d(f(x) - g(x))}{dx} = \dfrac{df(x)}{dx} - \dfrac{dg(x)}{dx}$

- $\dfrac{d(c\,f(x))}{dx} = c\dfrac{df(x)}{dx}$

- $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

- $\dfrac{d(u/v)}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

- $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Calculus Formulas using Modern Functional Notation

Derivatives:

- $f'(a)$ is the derivative of the function $f(x)$ at $x = a$.

- $\sin'(x) = \cos x$

- $\cos'(x) = -\sin x$

- $\ln'(x) = \dfrac{1}{x}$

Differentiation Rules:

- $(f + g)'(a) = f'(a) + g'(a)$

- $(f - g)'(a) = f'(a) - g'(a)$

- $(c\,f)'(a) = c\,f'(a)$

- $(f\,g)'(a) = g(a)\,f'(a) + f(a)\,g'(a)$

- $\left(\dfrac{f}{g}\right)'(a) = \dfrac{g(a)\,f'(a) - f(a)\,g'(a)}{(g(a))^2}$

- $(f \circ g)'(a) = f'(g(a))\,g'(a)$

Review: The Tangent Line Problem

This is one of the two classic geometric problems which historically motivated the development of calculus. It can be stated as:

Given the graph of a non-linear function, $y = f(x)$, what is the slope of the curve at the point $(a, f(a))$?

One solution is to consider two points on the curve, $P = (a, f(a))$, the point in question, and $Q = (a+h, f(a+h))$, for $h \neq 0$, another point on the graph of f. The line joining P and Q is called a secant line and its slope is

$$\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}.$$
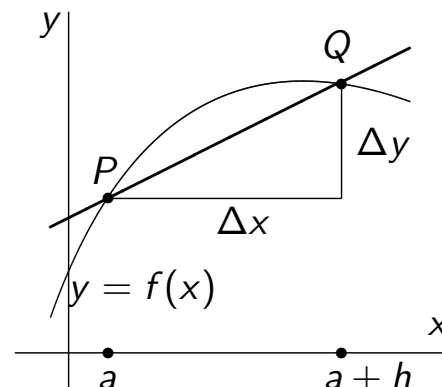
Since $\Delta y / \Delta x$ is undefined if $h = 0$, we consider what happens to the value of $\Delta y / \Delta x$ as h approaches 0, which we write as $h \to 0$.

Slope of a Tangent Line

Geometrically the limiting position of the secant lines joining P and Q as $h \to 0$ is called the tangent line to $y = f(x)$ at $x = a$; and algebraically, the limit of $\Delta y / \Delta x$ as $h \to 0$, written as

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

is the slope of the tangent line to the curve $y = f(x)$ at $x = a$. It is this limit that is considered to be the slope of $y = f(x)$ at $x = a$.

But exactly what is a limit? and how to calculate one? There are many answers to these questions, some more formal and theoretical than others. In this course we use what can be called an informal, or intuitive, approach to limits, making use of various techniques that you should have seen in high school.

Example 1

Find the slope, m, of the curve $f(x) = x^2$ at $a = -1$.
Solution: in an *intuitive* approach to limits you manipulate the expression

$$\frac{f(-1+h) - f(-1)}{h},$$

using techniques that you have probably already seen, assuming $h \neq 0$, until it makes sense to substitute $h = 0$. See calculations shown to the right. So the slope of $f(x)$ at $a = -1$ is -2. If the

$$
\begin{aligned}
m &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h \to 0} \frac{(h-1)^2 - (-1)^2}{h} \\
&= \lim_{h \to 0} \frac{h^2 - 2h}{h}, \text{ simplifying} \\
&= \lim_{h \to 0} (h - 2), \text{ if } h \neq 0 \\
&= -2, \text{ let } h = 0
\end{aligned}
$$

apparent logical contradiction—first $h \neq 0$, then $h = 0$—in the above approach bothers you, you are not alone. However, that's just the way it is with our intuitive, informal approach.

Graphs for Example 1, and Some Comments
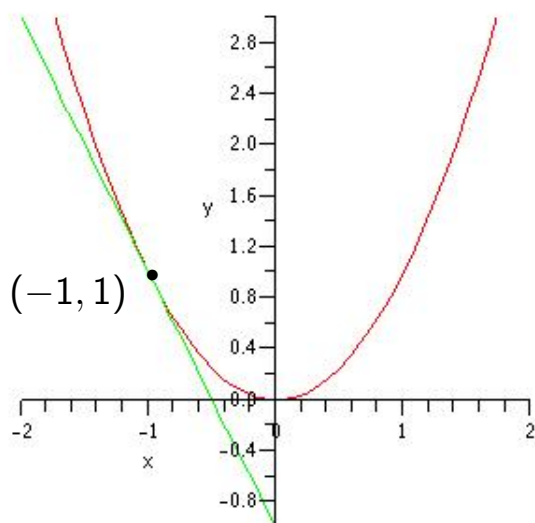
y=x^2
y=-2x-1

▶ The tangent line to the graph of $f(x) = x^2$ at the point $(-1, 1)$ has equation $y = -2x - 1$, as you can check.

▶ As you should know,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is called the derivative of $f(x)$ at $x = a$, and is written as $f'(a)$, or as

$$\left. \frac{df(x)}{dx} \right|_{x=a}.$$

More Review: Average Velocity

The tangent line problem can be given a physical interpretation in terms of velocity. For example, if $s(t)$ is the position of a particle at time t, then the average velocity of the particle on the time interval $a \leq t \leq b$ is given by

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}.$$

This corresponds to the geometric problem of finding the slope between the two points $(a, s(a))$ and $(b, s(b))$. On the other hand, the instantaneous velocity of the particle is its velocity at a given specific instant. This is harder to calculate, and corresponds to calculating the slope of a curve at a particular point.

Instantaneous Velocity

The formula for the instantaneous velocity of a particle at $t = a$ is given, not surprisingly, by

$$v_{inst} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

If you let $t = a + h$, then $\Delta t = a + h - a = h$, and

$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h},$$

which is just the slope of the tangent line to the graph of $s(t)$ at $t = a$. Moreover, in terms of derivatives you can say that

$$v_{inst} = \frac{ds}{dt}\bigg|_{t=a} \quad \text{or} \quad s'(a).$$

Example 2

Let $s = 4t^2 - 16t + 10$. The average velocity of the particle over the interval $1 \leq t \leq 2$ is:

$$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{-6 - (-2)}{1} = -4.$$

Its instantaneous velocity at $t = 1$ is given by

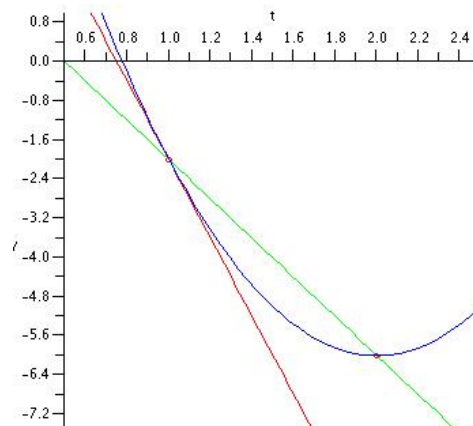$$\left.\frac{ds}{dt}\right|_{t=1} \quad \text{or } s'(1).$$

As you may recall from high school,

$$s'(t) = 8t - 16,$$

so $s'(1) = -8$.

Graphs for Example 2

- the blue curve is the graph of $s(t)$;
- the slope of the green line is the average velocity on the interval $1 \leq t \leq 2$: -4.
- the slope of the red line is the instantaneous velocity at $t = 1$: -8.

Aside: calculating average velocities requires more work than calculating instantaneous velocities, assuming you know how to calculate derivatives!

The Area Problem

The tangent line problem is one of the geometric problems that historically motivated calculus. The other problem is the area problem. It can be stated as:

Suppose $f(x) \geq 0$ for $a \leq x \leq b$. What is the area of the region below the graph of $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$?
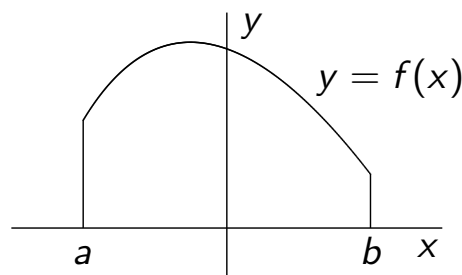

To solve this problem, we shall only assume that the area of a rectangle is known: width times length.

Some useful notation: as intervals of the form $a \leq x \leq b$ frequently show up in what follows, we shall denote this interval by $[a, b]$. That is, $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Geometric Solution to the Area Problem

One way to approximate the area of the region under a curve is to use lots of rectangles to approximate it. This is illustrated in the figures below, in which first 10 rectangles, then 50 rectangles are used to approximate the area under $y = 6 + 3\sin^2(x) + 4\cos^3(x)$ on the interval $[0, \pi]$. Note that as the number of rectangles increases, the total area of all the rectangles appears to become a better approximation to the actual area A under the curve.
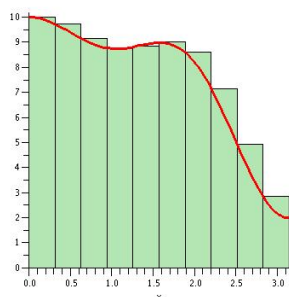
$A \approx 24.81858$
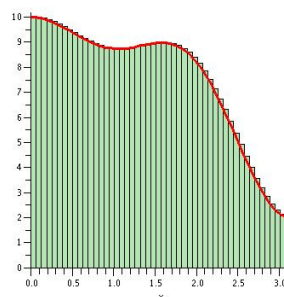
$A \approx 23.81327$

The Area Problem and Limits

The calculations used to approximate A in the previous slide were

$$A \approx \sum_{n=0}^{9} \left(6 + 3\sin^2\left(\frac{\pi n}{10}\right) + 4\cos^3\left(\frac{\pi n}{10}\right) \right) \frac{\pi}{10} \approx 24.81858$$

for 10 rectangles, and

$$A \approx \sum_{n=0}^{49} \left(6 + 3\sin^2\left(\frac{\pi n}{50}\right) + 4\cos^3\left(\frac{\pi n}{50}\right) \right) \frac{\pi}{50} \approx 23.81327$$

for 50 rectangles. These calculations are tedious enough, but the solution to the area problem, using limits, would require this calculation:

$$A = \lim_{n \to \infty} \sum_{i=0}^{n-1} \left(6 + 3\sin^2\left(\frac{\pi i}{n}\right) + 4\cos^3\left(\frac{\pi i}{n}\right) \right) \frac{\pi}{n},$$

which is truly an intimidating calculation!   (FYI: $A = 7.5\pi$)

What Is Calculus About?

▶ The concept of limits is what distinguishes calculus from other branches of mathematics.

▶ The solution to the tangent line problem involves one type of limit, called a derivative. Much of MAT186H1F will involve calculations and applications of derivatives; this branch of calculus is know as *differential calculus.*

▶ The solution to the area problem involves another type of limit, called a definite integral. The calculation and application of definite integrals is known as *integral calculus.*

▶ For those interested, the exact value of the area in the example of the previous slide is given by a definite integral:

$$A = \int_0^\pi \left(6 + 3\sin^2(x) + 4\cos^3(x) \right) \, dx = \frac{15\pi}{2}.$$