MAT186H1F Lec0103 Burbulla

Chapter 6 Lecture Notes

Fall 2011

Chapter 6: Applications of The Definite Integral

Integral Formulas

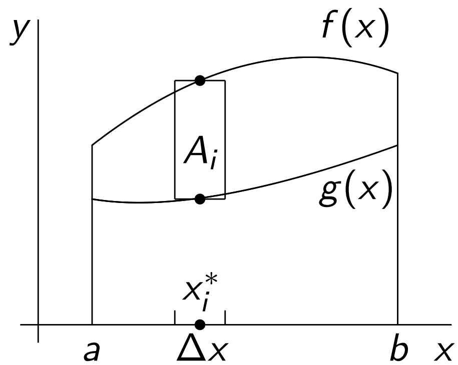
- Often to find a formula to describe something – be it geometrical or physical – we use a Riemann sum to first set things up in a simple, approximate way.
- These approximations usually start with a regular partition.
- Then as the limit of the norm of the partition goes to zero, the Riemann sum approaches an integral.
- In Chapter 6, formulas for area, volumes, length, surface area, work – to name a few – will be derived this way.

Area Between Two Curves: The Basic Formula

If $f(x) \geq g(x)$ for $x \in [a, b]$, then

$$\int_a^b (f(x) - g(x))\,dx$$

is the area of the region below f, above g, for $a \leq x \leq b$.



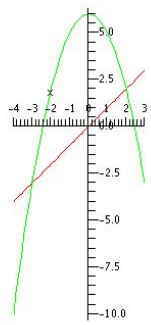
Proof:

$$A_i = (f(x_i^*) - g(x_i^*))\Delta x \Rightarrow A \simeq \sum A_i = \sum (f(x_i^*) - g(x_i^*))\Delta x;$$

$$A = \lim_{\Delta x \to 0} \sum A_i = \lim_{\Delta x \to 0} \sum (f(x_i^*) - g(x_i^*))\Delta x = \int_a^b (f(x) - g(x))\,dx.$$

Example 1

Find the area of the region bounded by $y = x$ and $y = 6 - x^2$.
Solution: find intersection points:
$6 - x^2 = x \Leftrightarrow 0 = x^2 + x - 6 \Leftrightarrow x = -3$ or $x = 2$.



$$\begin{aligned} A &= \int_{-3}^{2} \left(6 - x^2 - x\right)\, dx \\ &= \left[6x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-3}^{2} \\ &= 12 - \frac{8}{3} - \frac{4}{2} + 18 - \frac{27}{3} + \frac{9}{2} \\ &= \frac{125}{6} \end{aligned}$$

Example 2

Find the area between the x-axis and the curve $y = \sin(2x) + \sin x$ on $[0, \pi]$. **Solution:** $2 \sin x \cos x + \sin x = 0 \Rightarrow x = 0, \pi, 2\pi/3$.

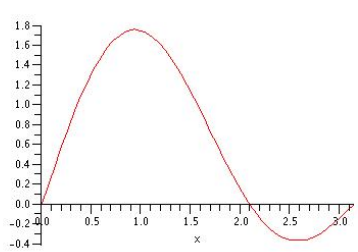
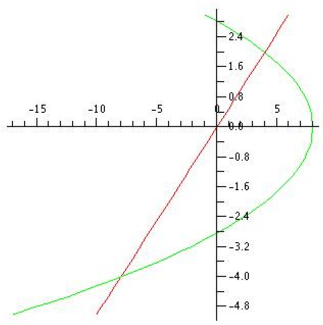


Figure: $A = A_1 + A_2$

$$\begin{aligned} A_1 &= \int_{0}^{2\pi/3} \left(\sin(2x) + \sin x\right)\, dx \\ &= \left[-\frac{1}{2}\cos(2x) - \cos x\right]_{0}^{2\pi/3} = \frac{9}{4} \\ A_2 &= \int_{2\pi/3}^{\pi} \left(-\sin(2x) - \sin x\right)\, dx \\ &= \left[\frac{1}{2}\cos(2x) + \cos x\right]_{2\pi/3}^{\pi} = \frac{1}{4} \end{aligned}$$

Example 3: Integrating with Respect to y

Find area of region bounded by the curves with equation $2y = x$ and $y^2 = 8 - x$. **Solution:** find intersection points: $y^2 = 8 - 2y \Leftrightarrow y^2 + 2y - 8 = 0 \Leftrightarrow y = -4$ or $y = 2$.

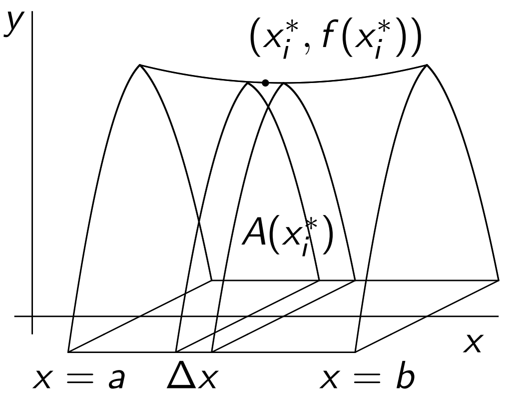


$$\begin{aligned} A &= \int_{-4}^{2} \left(8 - y^2 - 2y\right) \, dy \\ &= \left[8y - \frac{1}{3}y^3 - y^2\right]_{-4}^{2} \\ &= 16 - \frac{8}{3} - 4 + 32 - \frac{64}{3} + 16 \\ &= 36 \end{aligned}$$

Example 3, Integrating with Respect to x; the Hard Way

$$\begin{aligned} A &= \int_{-8}^{4} \left(\frac{x}{2} - (-\sqrt{8-x})\right) \, dx + \int_{4}^{8} \left(\sqrt{8-x} - (-\sqrt{8-x})\right) \, dx \\ &= \int_{-8}^{4} \left(\frac{x}{2} + \sqrt{8-x}\right) \, dx + \int_{4}^{8} 2\sqrt{8-x} \, dx \\ &= \left[\frac{x^2}{4} - \frac{2(8-x)^{3/2}}{3}\right]_{-8}^{4} + \left[-\frac{4(8-x)^{3/2}}{3}\right]_{4}^{8} \\ &= 4 - \frac{16}{3} - 16 + \frac{128}{3} - 0 + \frac{32}{3} \\ &= 36, \text{ as before.} \end{aligned}$$

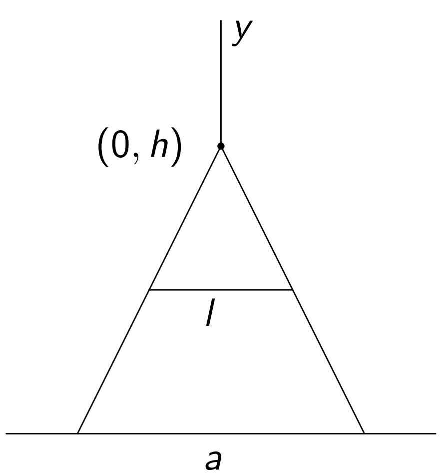
The Method of Slicing



Let x_i^* be any point in a subinterval of length Δx. Let the area of a cross-sectional slice perpendicular to the x-axis with base Δx be $A(x_i^*)$. To find the volume of the solid we add up the volumes of the slices:

$$\begin{aligned} V &\simeq \sum \Delta V = \sum A(x_i^*)\,\Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum A(x_i^*)\,\Delta x \\ &= \int_a^b A(x)\,dx \end{aligned}$$

Example 1: Volume of a Pyramid, $V = \frac{1}{3}a^2 h$

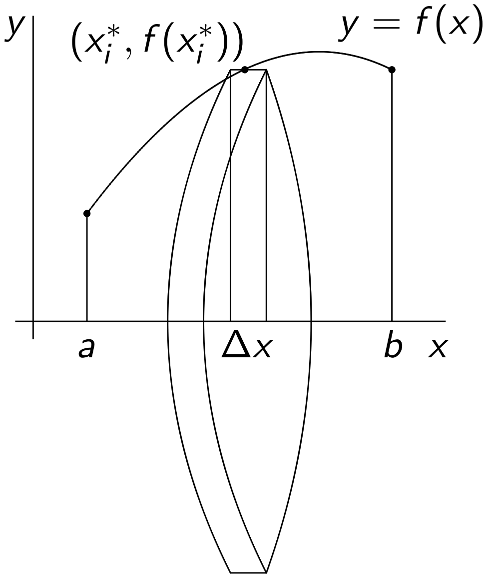


Side view

Let l be the length of the side of a cross-section of the pyramid at height y above the base. By similar triangles, $\dfrac{h}{a} = \dfrac{h-y}{l} \Leftrightarrow l = a\left(1 - \dfrac{y}{h}\right)$. So

$$\begin{aligned} V &= \int_0^h A(y)\,dy = \int_0^h a^2 \left(1 - \frac{y}{h}\right)^2 dy \\ &= a^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy \\ &= a^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = \frac{1}{3}a^2 h \end{aligned}$$

Solids of Revolution: Method of Disks



Let x_i^* be any point in a subinterval of length Δx. The volume of the disc obtained by revolving about the x-axis the rectangle with base Δx and radius $f(x_i^*)$ is $\Delta V = \pi f(x_i^*)^2\,\Delta x$.

$$\begin{aligned} V &\simeq \sum \Delta V = \sum \pi f(x_i^*)^2\,\Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum \pi f(x_i^*)^2\,\Delta x \\ &= \int_a^b \pi f(x)^2\,dx \end{aligned}$$

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6.1 Area Between Two Curves
6.2 Volumes by Slicing: Disks and Washers
6.3 Volumes by Cylindrical Shells
6.4 Length of a Plane Curve
6.5 Area of a Surface of Revolution
6.6 Work
6.9 Hyperbolic Functions: An Overview

Example 2: Volume of a Sphere

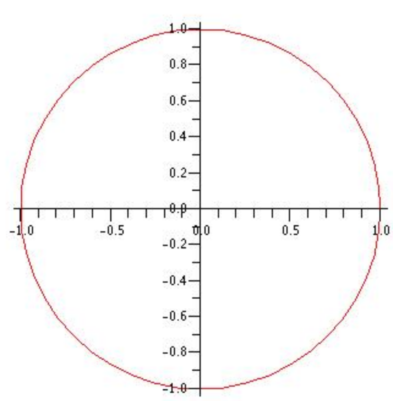

Figure: $x^2 + y^2 = r^2$

$$\begin{aligned} V &= 2\int_0^r \pi y^2\,dx \\ &= 2\pi \int_0^r (r^2 - x^2)\,dx \\ &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi \left(r^3 - \frac{r^3}{3} \right) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

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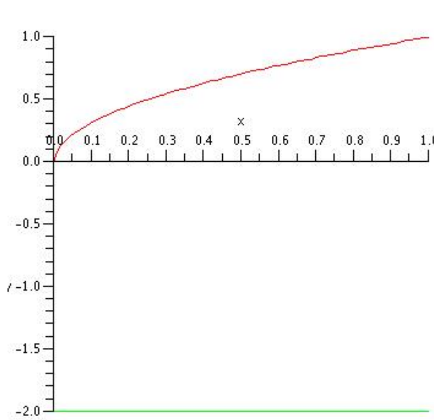
Example 3

Find the volume of the solid obtained by revolving around the line $y = -2$ the curve $y = \sqrt{x}$ for $x \in [0, 1]$.



$$
\begin{aligned}
V &= \int_0^1 \pi(\sqrt{x}+2)^2\,dx \\
&= \pi \int_0^1 (x + 4\sqrt{x} + 4)\,dx \\
&= \pi \left[\frac{x^2}{2} + \frac{8}{3}x^{3/2} + 4x\right]_0^1 \\
&= \pi\left(\frac{1}{2} + \frac{8}{3} + 4\right) = \frac{43}{6}\pi
\end{aligned}
$$

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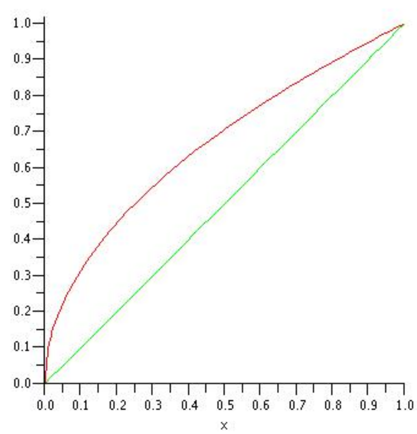
Example 4

Find the volume of the solid obtained by revolving about the x-axis the region bounded by the curves $y = \sqrt{x}$ and $y = x$ for $x \in [0, 1]$.



$$
\begin{aligned}
V &= \int_0^1 \pi(r_o^2 - r_i^2)\,dx \\
&= \pi \int_0^1 (x - x^2)\,dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\
&= \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\pi
\end{aligned}
$$

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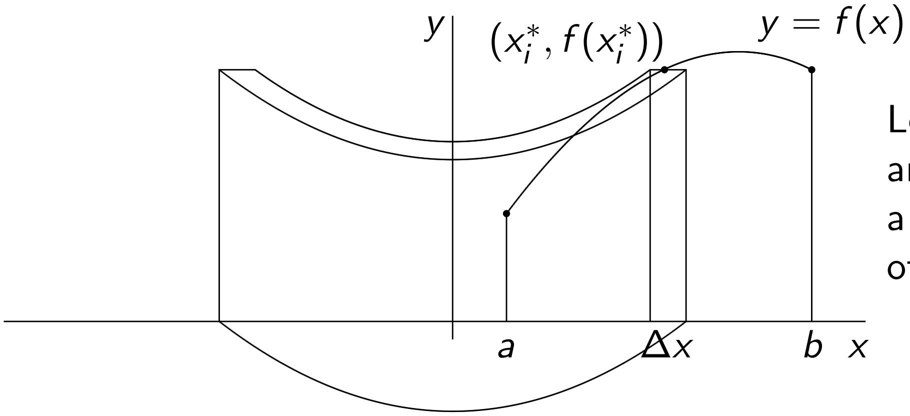
Suppose $y = f(x)$ on $[a, b]$ is Revolved Around the y-axis



Let x_i^* be any point in a subinterval of length Δx.

The approximate volume of the above cylindrical shell is $\Delta V \simeq C \cdot h \cdot w = 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x$.

Volumes By Cylindrical Shells

Then

$$\begin{aligned} V &\simeq \sum \Delta V \\ &= \sum 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x \\ &= \int_a^b 2\pi x \cdot f(x)\, dx \end{aligned}$$

Example 1: Volume of a Cone

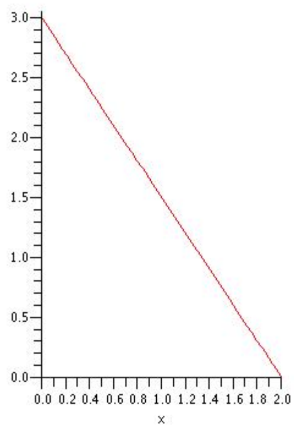


Figure: $\frac{x}{r} + \frac{y}{h} = 1$

$$\begin{aligned}
V &= \int_0^r 2\pi x\, h\left(1 - \frac{x}{r}\right)\, dx \\
&= 2\pi h \int_0^r \left(x - \frac{x^2}{r}\right)\, dx \\
&= 2\pi h \left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r \\
&= 2\pi h \left(\frac{r^2}{2} - \frac{r^2}{3}\right) \\
&= \frac{1}{3}\pi h r^2
\end{aligned}$$

Example 2: Volume of a Sphere

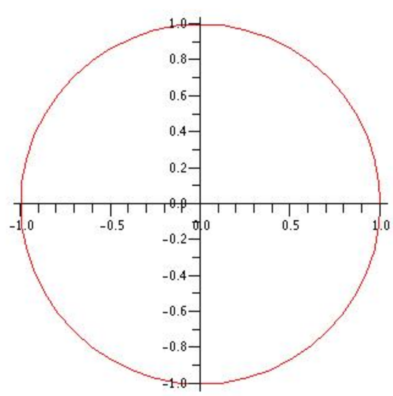
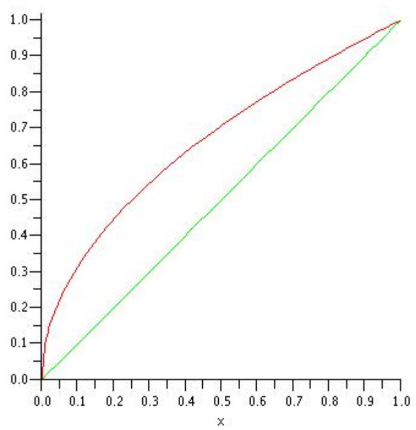


Figure: $x^2 + y^2 = r^2$

$$\begin{aligned}
V &= 2\int_0^r 2\pi x \cdot y\, dx \\
&= 2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx \\
&= 2\pi \int_{r^2}^0 \sqrt{u}\,(-du) \\
&= 2\pi \int_0^{r^2} \sqrt{u}\, du \\
&= 2\pi \left[\frac{2}{3}u^{3/2}\right]_0^{r^2} = \frac{4}{3}\pi r^3
\end{aligned}$$

Example 3

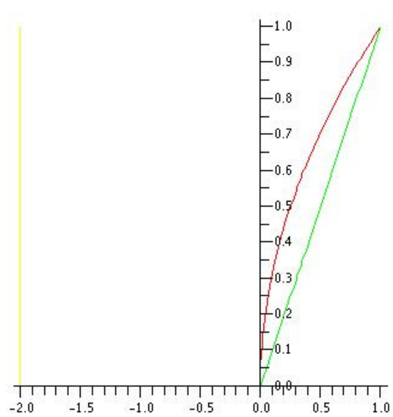
Find the volume of the solid obtained by revolving around the y-axis the region bounded by $y = \sqrt{x}$ and $y = x$, for $x \in [0, 1]$.



$$\begin{aligned} V &= \int_0^1 2\pi x\,(\sqrt{x} - x)\,dx \\ &= 2\pi \int_0^1 (x^{3/2} - x^2)\,dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^3}{3}\right]_0^1 \\ &= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2}{15}\pi \end{aligned}$$

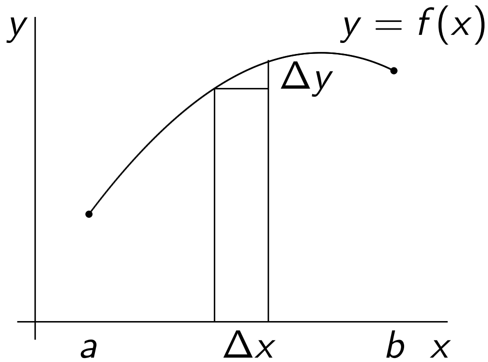
Example 4

Find the volume of the solid obtained by revolving around the line $x = -2$ the region bounded by $y = \sqrt{x}$ and $y = x$, for $x \in [0, 1]$.



$$\begin{aligned} V &= \int_0^1 2\pi(x+2)\,(\sqrt{x} - x)\,dx \\ &= 2\pi \int_0^1 (x^{3/2} + 2\sqrt{x} - x^2 - 2x)\,dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} - \frac{x^3}{3} - x^2\right]_0^1 \\ &= 2\pi\left(\frac{2}{5} + \frac{4}{3} - \frac{1}{3} - 1\right) = \frac{4}{5}\pi \end{aligned}$$

The Length of a Curve



If Δs is the length of a small part of the curve, then

$$\Delta s^2 \simeq \Delta x^2 + \Delta y^2$$

$$\Rightarrow \Delta s \simeq \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

Now let $\Delta x \to 0$:

$$S = \lim_{\Delta x \to 0} \sum \Delta s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

S is called the arc length, or simply length, of the curve.

Example 1

The expression $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ is called the arc length differential. Then $S = \int_a^b ds$. **Example:** let $f(x) = x^{2/3}$ on $[0, 1]$.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2}\, dx = \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx$$

$$\Rightarrow S = \int_0^1 ds = \int_0^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx = \frac{1}{18}\int_4^{13} \sqrt{u}\, du$$

$$(u = 9x^{2/3} + 4) = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}(13^{3/2} - 8)$$

Example 2: Circumference of a Circle

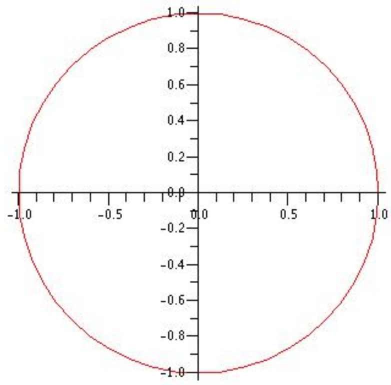


Figure: $x^2 + y^2 = r^2$

$$\begin{aligned} C &= 4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 4\int_0^r \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx \\ &= 4\int_0^r \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx \\ &= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx, \text{ since } r>0 \end{aligned}$$

Example 2, Continued

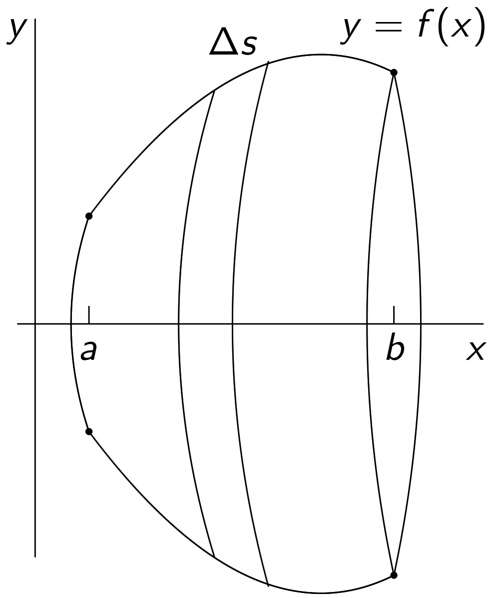
To evaluate this integral, we shall use inverse trigonometric functions. In particular:

$$\int \frac{dx}{\sqrt{r^2-x^2}} = \sin^{-1}\left(\frac{x}{r}\right) + C,$$

as you may check. Then

$$\begin{aligned} C &= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx \\ &= 4r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r \\ &= 4r(\sin^{-1}(1)-\sin^{-1}(0)) \\ &= 2\pi r, \text{ as you may check.} \end{aligned}$$

Surface Area



Partition the interval $[a, b]$ into n subintervals of equal length Δx. In each subinterval pick a value x_i^* and consider the strip of surface obtained by revolving the curve $y = f(x)$ on the interval $[x_{i-}, x_1]$ around the x-axis. It's approximate radius is $r_i = f(x_i^*)$, and its approximate width is Δs. So the approximate area of the strip is

$$\Delta A = 2\pi f(x_i^*)\Delta s.$$

Surface Area Formulas

Then the surface area of the solid of revolution obtained by revolving the curve $y = f(x)$ on the interval $[a, b]$ around the x-axis is given by

$$SA = \lim_{\Delta x \to 0} \sum \Delta A = \int_a^b 2\pi f(x)\, ds = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If instead you revolve the curve around the y-axis, then the approximate radius of the strip becomes x_i^* and so the formula for the surface area becomes

$$SA = \int_a^b 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example 2: Surface Area of a Sphere is $4\pi r^2$

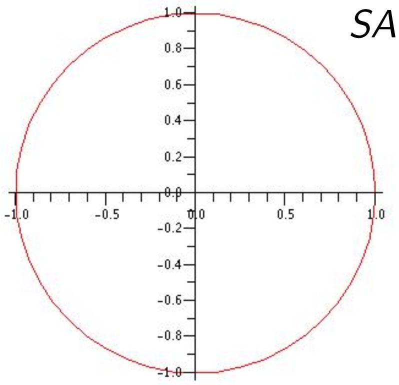


Figure: $x^2 + y^2 = r^2$

$$
\begin{aligned}
SA &= 2\int_0^r 2\pi y \, ds \\
&= 4\pi \int_0^r \sqrt{r^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} \, dx \\
&= 4\pi \int_0^r \sqrt{r^2 - x^2 + x^2} \, dx \\
&= 4\pi \int_0^r r \, dx, \text{ since } r > 0 \\
&= 4\pi r [x]_0^r = 4\pi r^2
\end{aligned}
$$

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Example 3: Surface Area of a Cone is $\pi r\sqrt{r^2 + h^2}$

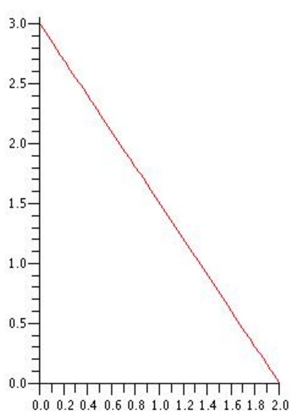

Figure: $\dfrac{x}{r} + \dfrac{y}{h} = 1$

$$
\begin{aligned}
SA &= \int_0^r 2\pi x \, ds \\
&= 2\pi \int_0^r x\sqrt{1 + \left(\frac{-h}{r}\right)^2} \, dx \\
&= 2\pi \int_0^r x \frac{\sqrt{r^2 + h^2}}{r} \, dx \\
&= 2\pi \frac{\sqrt{r^2 + h^2}}{r} \left[\frac{x^2}{2}\right]_0^r \\
&= \pi r \sqrt{r^2 + h^2}
\end{aligned}
$$

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Work in Physics

The work required to move an object through a distance d by applying a constant force F is given by

$$W = F\,d.$$

Suppose an object is moved from $x = a$ to $x = b$ by applying a non-constant force $F(x)$. You can approximate the work on a small subinterval of length Δx, by picking a point x_i^* in the subinterval and taking the force to be constant, $F(x_i^*)$, over that subinterval. Then on each subinterval the work done is $\Delta W \simeq F(x_i^*)\Delta x$. The total work done is

$$W = \lim_{\Delta x \to 0} \sum \Delta W = \int_a^b F(x)\,dx.$$

Two Formulas from Physics

To use the formula $W = \int_a^b F(x)\,dx$ you have to know the force in terms of x. Here are two examples in which such a force is known:

1. Hooke's Law: if you stretch a mass on a spring the force required is proportional to the displacement. That is $F(x) = kx$, where $x = 0$ is the equilibrium position of the spring.
2. Newton's Law of Gravity: if m_1 and m_2 are separated by a distance x then the gravitational force of attraction between the two masses is given by

$$F = \frac{Gm_1m_2}{x^2},$$

where G is the gravitational constant.

Example 1

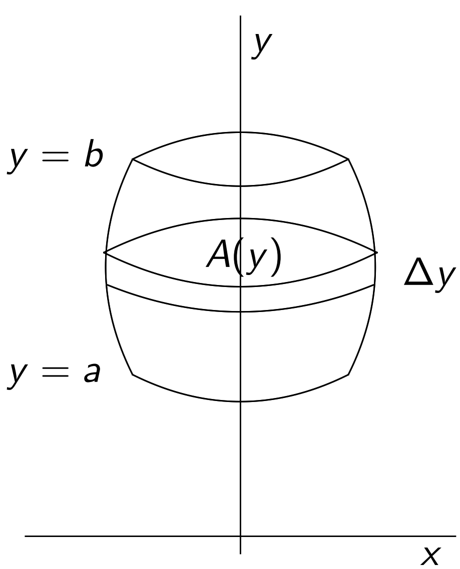
1. The work done to stretch a spring from $x = a$ to $x = b$ is

$$W = \int_a^b kx\,dx = k\left[\frac{x^2}{2}\right]_a^b = \frac{k}{2}(b^2 - a^2).$$

2. If R is the radius of the earth, and m_2 its mass, then the work required to put a satellite of mass m_1 into an orbit of height h above the earth's surface is

$$\begin{aligned} W &= \int_R^{R+h} \frac{Gm_1 m_2}{x^2}\,dx = \left[-\frac{Gm_1 m_2}{x}\right]_R^{R+h} \\ &= -\frac{Gm_1 m_2}{R+h} + \frac{Gm_1 m_2}{R} = \frac{Gm_1 m_2 h}{R(R+h)} \end{aligned}$$

Work Done in Filling a Tank



Suppose a fluid of density ρ is pumped from ground level $y = 0$ up into a tank, with base at $y = a$ and top at $y = b$. Suppose the cross-sectional area of the tank at height y is $A(y)$. Consider a thin shell of the fluid of thickness Δy. The volume of this shell is approximately $\Delta V = A(y)\,\Delta y$. Its mass is approximately $\rho\Delta V$ and the work required to pump this thin shell of liquid up to height y is approximately $\Delta W = \rho\Delta V\,g\,y$.

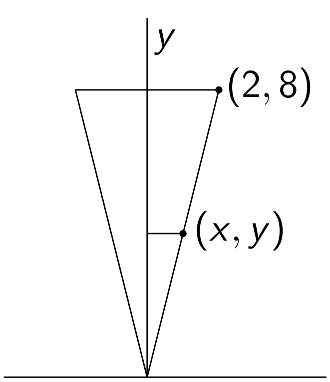
Formulas for Work Done in Filling or Emptying a Tank

1. Thus the work required to pump the tank full of fluid is
$$W = \lim_{\Delta y \to 0} \sum \Delta W = \lim_{\Delta y \to 0} \sum \rho A(y) \Delta y \, g \, y = \lim_{\Delta y \to 0} \sum \rho g \, A(y) \, y \, \Delta y = \int_a^b \rho g \, A(y) \, y \, dy$$
2. If the tank is emptied by pumping all the liquid up to a pipe or conduit above the tank at height $y = h$ then the work done in emptying the tank is
$$W = \int_a^b \rho g \, A(y) \, (h - y) \, dy.$$

Example 2

Find the work done in pumping fluid of density ρ from ground level into a conical tank with radius at the top 2 m and height 8 m.

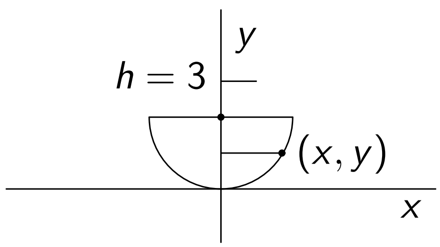
By similar triangles, $\dfrac{x}{y} = \dfrac{2}{8} \Leftrightarrow x = \dfrac{1}{4} y.$



$$A(y) = \pi x^2 = \frac{\pi}{16} y^2$$
$$\Rightarrow W = \int_0^8 \rho g \, \frac{\pi}{16} y^2 \, y \, dy = \frac{\pi \rho g}{16} \int_0^8 y^3 \, dy = \frac{\pi \rho g}{16} \left[\frac{y^4}{4} \right]_0^8 = 64 \pi \rho g \text{ (Joules)}$$

Example 3

A hemispherical tank of radius 2 m is full of liquid with density ρ. How much work is required to empty the tank by pumping all the liquid up to a pipe 1 m above the top of the tank?



$x^2 + (y-2)^2 = 4$

$$\begin{aligned} A(y) &= \pi x^2 \\ &= \pi\left(4-(y-2)^2\right) \\ &= \pi(4-y^2+4y-4) \\ &= \pi(4y-y^2) \end{aligned}$$

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Example 3, Concluded

$$\begin{aligned} W &= \int_a^b \rho g\, A(y)\,(h-y)\,dy \\ &= \int_0^2 \rho g\, \pi(4y-y^2)\,(3-y)\,dy \\ &= \int_0^2 \rho g \pi(y^3-7y^2+12y)\,dy \\ &= \rho g \pi \left[\frac{y^4}{4} - \frac{7}{3}y^3 + 6y^2\right]_0^2 \\ &= \rho g \pi (4 - \frac{56}{3} + 24) = \frac{28}{3}\rho g \pi \end{aligned}$$

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The Work-Energy Relationship

Recall: $F = ma = m\frac{dv}{dt}$. Suppose an object of mass m is moved by a force F from $x = a$ at time $t = t_i$ to $x = b$ at time $t = t_f$. Then

$$\begin{aligned} W &= \int_a^b F\,dx = \int_a^b m\frac{dv}{dt}\,dx \\ &= \int_{t_i}^{t_f} m\frac{dv}{dt}\frac{dx}{dt}\,dt = \int_{t_i}^{t_f} m\,v\,\frac{dv}{dt}\,dt \\ \text{(by substitution)} &= \int_{v_i}^{v_f} m\,v\,dv = \left[\frac{mv^2}{2}\right]_{v_i}^{v_f} \\ &= \frac{mv_f^2}{2} - \frac{mv_i^2}{2}, \text{ the change in kinetic energy} \end{aligned}$$

Example 4

A mass of 10 kg is moving along the x-axis with speed 5 m/sec. At position $x = 0$ a force $F(x) = 3x^2$ N begins to push the object. What is the speed of the object when it reaches $x = 10$? Assume position along the x-axis is measured in meters.

Solution:

$$W = \int_0^{10} F(x)\,dx = \int_0^{10} 3x^2\,dx = \left[x^3\right]_0^{10} = 1000.$$

$$\frac{10v_f^2}{2} - \frac{10v_i^2}{2} = 1000 \Leftrightarrow 5v_f^2 = 1000 + 125 \Leftrightarrow v_f^2 = 225 \Leftrightarrow v_f = 15$$

Two Different Trigonometries

As opposed to the six regular trigonometric functions,

$$\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta \text{ and } \cot\theta,$$

which can be defined in terms of the circle $x^2 + y^2 = 1$, the six hyperbolic trigonometric functions are defined in terms of the hyperbola $x^2 - y^2 = 1$. However, this is not apparent from their definitions:

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}, \cosh\theta = \frac{e^\theta + e^{-\theta}}{2},$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \operatorname{csch}\theta = \frac{1}{\sinh\theta}, \operatorname{sech}\theta = \frac{1}{\cosh\theta}, \coth\theta = \frac{\cosh\theta}{\sinh\theta}.$$

Basic Hyperbolic Trigonometric Identity

$$\begin{aligned}\cosh^2\theta - \sinh^2\theta &= \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 - \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2 \\ &= \frac{e^{2\theta} + 2 + e^{-2\theta}}{4} - \frac{e^{2\theta} - 2 + e^{-2\theta}}{4} \\ &= \frac{4}{4} = 1.\end{aligned}$$

If $x = \cosh\theta$ and $y = \sinh\theta$, then the basic hyperbolic trig identity states that

$$x^2 - y^2 = 1,$$

which is the equation of an hyperbola.

Graphs of $\sinh\theta$ and $\cosh\theta$

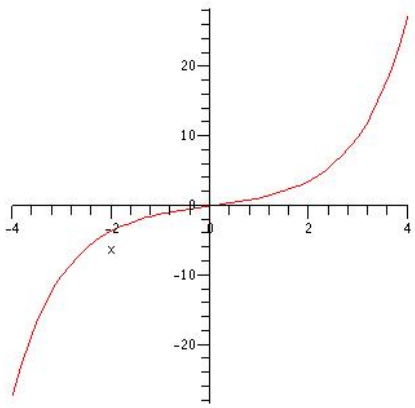

Figure: hyperbolic sine

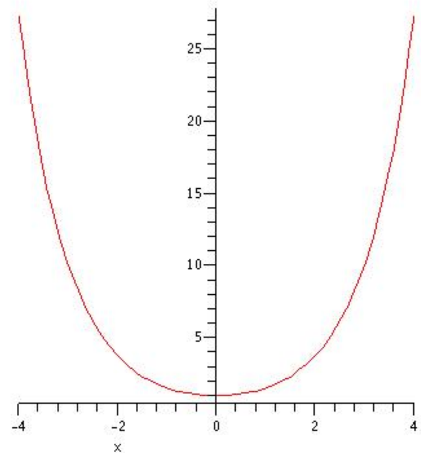

Figure: hyperbolic cosine

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Hyperbolic Trigonometric Indentities

The following identities can all be proved by putting all hyperbolic trig functions in terms of e^θ and $e^{-\theta}$ and simplifying:

1. $1 - \tanh^2\theta = \operatorname{sech}^2\theta$
2. $\coth^2\theta - 1 = \operatorname{csch}^2\theta$
3. $\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$
4. $\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$
5. $\sinh 2\theta = 2\sinh\theta\cosh\theta$
6. $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$
7. $\cosh^2\theta = \frac{1}{2}(\cosh(2\theta)+1)$
8. $\sinh^2\theta = \frac{1}{2}(\cosh(2\theta)-1)$

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Derivatives of $\sinh\theta$ and $\cosh\theta$

$$\begin{aligned}\sinh'\theta &= \frac{d}{d\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)\\ &= \frac{e^{\theta}+e^{-\theta}}{2}\\ &= \cosh\theta\end{aligned}$$

Similarly,

$$\cosh'\theta = \frac{d}{d\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) = \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta$$

Derivatives of the Other Hyperbolic Trig Functions

You can also prove that

1. $\tanh'\theta = \operatorname{sech}^2\theta$
2. $\coth'\theta = -\operatorname{csch}^2\theta$
3. $\operatorname{sech}'\theta = -\operatorname{sech}\theta\tanh\theta$
4. $\operatorname{csch}'\theta = -\operatorname{csch}\theta\coth\theta$

Observe that all the formulas from the last few slides are very similar to the corresponding regular trig identities you are already familiar with, except for the odd plus or minus sign. In fact, you shouldn't memorize any of these formulas; you should simply be aware of what the hyperbolic trig functions are, just in case they show up in the homework, or in your other courses.

Inverse Hyperbolic Trig Functions

Since the hyperbolic trig functions are expressed in terms of exponentials, it should come as no surprise that the inverse hyperbolic trig functions can be expressed in terms of logarithms. For example,

$$\begin{aligned} y = \sinh^{-1} x \quad &\Leftrightarrow \quad x = \sinh y \\ &\Leftrightarrow \quad x = \frac{e^y - e^{-y}}{2} \\ &\Leftrightarrow \quad 2x = e^y - \frac{1}{e^y} \\ &\Leftrightarrow \quad 2xe^y = e^{2y} - 1 \\ (\text{ using the quadratic formula }) \quad &\Leftrightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \end{aligned}$$

Formula for $y = \sinh^{-1} x$

So far, we have:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$ for all y, we must take

$$e^y = x + \sqrt{x^2 + 1},$$

from which we get

$$\sinh^{-1} x = y = \ln(x + \sqrt{x^2 + 1}).$$

There are similar formulas for the other five inverse hyperbolic trig functions.

Derivatives of Inverse Hyperbolic Trig Functions

These can be calculated implicitly, or directly.

$$\begin{aligned} y = \sinh^{-1} x \quad &\Rightarrow \quad \sinh y = x \\ &\Rightarrow \quad \cosh y \frac{dy}{dx} = 1 \\ &\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cosh y} \\ &\Rightarrow \quad \frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Alternately,

$$\frac{d \sinh^{-1} x}{dx} = \frac{d}{dx} \ln(x + \sqrt{x^2+1}) = \frac{1 + x/\sqrt{x^2+1}}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^2}}.$$

Six New Integration Formulas

1. $\displaystyle\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C$
2. $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \text{ if } |u| > 1$
3. $\displaystyle\int \frac{du}{1-u^2} = \tanh^{-1} u + C, \text{ if } |u| < 1$
4. $\displaystyle\int \frac{du}{1-u^2} = \coth^{-1} u + C, \text{ if } |u| > 1$
5. $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\text{sech}^{-1} |u| + C, \text{ if } |u| < 1$
6. $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\text{csch}^{-1} |u| + C$