MAT186H1F Lec0103 Burbulla

General Advice, Trigonometry and Conic Sections

Fall 2011

Notation

- Mathematics has its own set of symbols. If you use them, you must use them correctly.
- You will lose marks on tests or exams if your notation is incorrect.
- Following are some examples of common errors.

Example 1

Suppose you are differentiating the function $y = x^2$. If you write

$$\begin{aligned} y &= x^2 \\ &= 2x \end{aligned}$$

it is incorrect, even though most people would know what you are doing. Of course, you should write

$$\begin{aligned} y &= x^2 \\ \Rightarrow y' &= 2x \end{aligned}$$

Or you could simply write

$$\frac{dx^2}{dx} = 2x.$$

Example 2

Don't confuse implication ($\Rightarrow$) with equality ($=$) . The symbol $\Rightarrow$ is a logical connective and means If … then … . For example,

$$\text{if } y = x^2, \text{ then } y' = 2x,$$

can be written as

$$y = x^2 \Rightarrow y' = 2x.$$

But writing something like

$$\begin{aligned} ax^2 + bx + c &\Rightarrow 0 \\ x &\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

is abuse of notation.

Example 3

The expressions $\frac{0}{0}, \frac{\infty}{\infty}, 0^0, 0 \cdot \infty, 1^\infty, \infty^0$ and $\infty - \infty$ are indeterminate. When evaluating limits of this type, which may or may not exist, don't *equate* the limit to one of the above expressions. For example, writing

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{0}{0}$$

is incorrect; the limit is actually equal to $\frac{3}{2}$. You *can* say the limit is of the form $\frac{0}{0}$; or that when $x = 1$, $\frac{x^3 - 1}{x^2 - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$, which is indeterminate.

Example 4

When reducing a matrix, don't put equal ($=$) signs between matrices which *aren't* equal:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

is incorrect. Instead, use an arrow ($\rightarrow$) between row equivalent matrices:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}.$$

Example 5

An indefinite integral always includes an arbitrary constant! To write

$$\int x^2\, dx = \frac{x^3}{3}$$

is incomplete, and will cost you a mark everytime. Correct is:

$$\int x^2\, dx = \frac{x^3}{3} + c.$$

Example 6

The correct indefinite integral of $\frac{1}{x}$ with respect to x is:

$$\int \frac{1}{x}\, dx = \ln |x| + c.$$

Without the absolute value it is wrong, and will cost you a mark everytime. Some students even write

$$\int \frac{1}{x}\, dx = \ln x,$$

which is doubly wrong – and will cost you *two* marks.

Logic

- Errors in logic that show up in your written solutions will cost you marks on a test or exam.
- They may even make your whole “solution” worthless – even if your calculations are correct.
- Some examples follow.

Example 7

The converse of a true statement is *not* necessarily a true statement. For example, it is true that if the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. But it is not necessarily true that if $\lim_{n\to\infty} a_n = 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ converges. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$. This is a very common misunderstanding in MAT187H1S, Calculus II.

Example 8

To show a statement is *false* you only have to exhibit one counterexample. For example, the statement

> *If the number of equations in a system of linear equations is less than the number of unknowns in the system, then the system has infinitely many solutions.*

is a false statement. Consider the example

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 1 \end{array}$$

which consists of two equations in three unknowns $(2 < 3)$ but has no solutions, because it is obviously inconsistent.

Example 9

To show a statement is *true*, it is not enough to illustrate it with one example. Consider the statement

If A is a 2×2 matrix such that $A^2 = I$, then $\det A = \pm 1$.

which *is* a true statement. You can't prove it is true by simply considering one choice of A. Instead, you must give a general argument:

$$\begin{aligned} A^2 = I \quad &\Rightarrow \quad \det\left(A^2\right) = \det I \\ &\Rightarrow \quad (\det A)^2 = 1 \\ &\Rightarrow \quad \det A = \pm 1 \end{aligned}$$

Inexcusable errors

- Should any of the following types of fundamental algebraic errors show up in one of your solutions, everything after the appearance of the error will be forfeited.
- In computer jargon, these errors can all be considered "fatal errors."
- The truth is that the following errors are very frequently made by first year students (who should know better), and as a result almost every first year math textbook now comes with some kind of algebra or trigonometry supplement, since it seems some students require them.

Example 10

$\sqrt{a^2} = a$ is wrong! Correct is: $\sqrt{a^2} = |a|$.

Example 11

$\sqrt{a^2 + b^2} = a + b$ is wrong! There just isn't any easy way to simplify this expression.

Example 12

$\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$ is wrong! Correct is

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

So,

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \Leftrightarrow (a+b)^2 = ab \Leftrightarrow a^2 + ab + b^2 = 0,$$

which is *impossible* if a and b are non-zero real numbers.

Example 13

$\ln(M+N) = \ln M + \ln N$ and $\ln\left(\dfrac{M}{N}\right) = \dfrac{\ln M}{\ln N}$ are both wrong!
There just isn't any easy way to simplify $\ln(M+N)$. Correct are:

$$\ln\left(\frac{M}{N}\right) = \ln M - \ln N \text{ and } \ln(MN) = \ln M + \ln N,$$

if both $M > 0$ and $N > 0$.

Example 14

$\sin(\alpha+\beta)=\sin\alpha+\sin\beta$ and $\sin(k\theta)=k\sin\theta$ are both wrong! Trigonometric functions are not linear. Correct is

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha.$$

There is no easy way to simplify $\sin(k\theta)$; but, for instance,

$$\sin(2\theta)=2\sin\theta\cos\theta$$

and

$$\sin(3\theta)=3\sin\theta-4\sin^3\theta.$$

Absolute Value

Probably the most misunderstood formula from high school is the definition of absolute value:

$$|a|=\begin{cases} a, & \text{if } a\geq 0\\ -a, & \text{if } a<0\end{cases}$$

It can be restated as

$$|a|=\sqrt{a^2},$$

which may or may not help you! But either way, make sure you understand this function!

Example 15

Thus $|x|^2 = x^2$ is correct, but $|x|^3 = x^3$ is incorrect. Can you explain why?

Some Comments About Marking

- If you make a copying error – or some other ‘dumb’ mistake – in your work which results in the rest of the problem being much easier than intended, then you will forfeit lots of marks.
- On the other hand, if you make a copying error and the rest of the problem remains of comparable difficulty, then you will only lose a few marks.
- An example follows.

Example 16

Consider the integral $\int \frac{1}{x^3+1}\,dx$, which is a fairly involved problem requiring partial fractions, completing the square and a trig substitution to solve. Say it is worth 10 marks on a test. If you miscopy the integral as $\int \frac{1}{x^3-1}\,dx$ you could still get 9 out of 10, if the rest of your work is correct, because the exact same procedures as the intended question are involved. However, if you miscopy the question as $\int \frac{1}{x^2+1}\,dx$ it becomes a one-liner, and you would get at most 1 mark out of 10.

A Word About Your Calculator

Don't blame your calculator if it gives you the wrong answer. For example, many calculators are programmed to accept only positive arguments when taking roots of a number. So, even though there is a real value for

$$(-1)^{(1/3)},$$

some caclulators will not evaluate this expression, or will display an answer of 0. The correct answer is of course

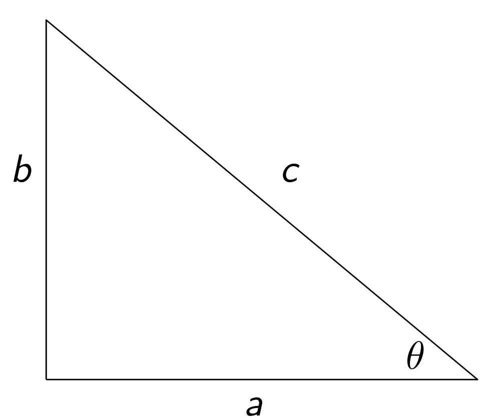
$$(-1)^{(1/3)} = -1.$$

Make sure you know how to use your calculator, and its built-in limits.

Introduction

1. Trigonometric functions can be defined in terms of right triangles or circles, as you should have seen in high school.
2. Hyperbolic trigonometric functions are defined in terms of an hyperbola, as we shall see in Section 6.9.
3. Hyperbolic trigonometric functions can also be defined in terms of exponential functions, as we shall see later.
4. Surprisingly, the regular trigonometric functions can also be defined in terms of exponential functions; you won't see this until your second calculus course, MAT187H1 Calculus II.

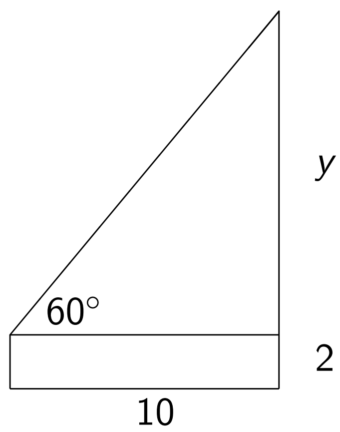
The Six Trigonometric Functions and Right Triangles



- $\sin\theta = \dfrac{b}{c}$
- $\csc\theta = \dfrac{c}{b} = \dfrac{1}{\sin\theta}$
- $\cos\theta = \dfrac{a}{c}$
- $\sec\theta = \dfrac{c}{a} = \dfrac{1}{\cos\theta}$
- $\tan\theta = \dfrac{b}{a} = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta}$

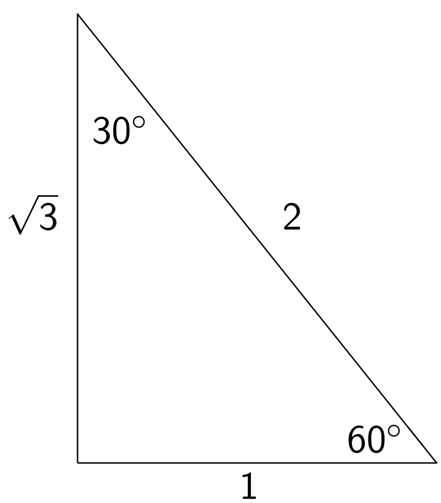
Example 1

Problem: At a certain point, 10 m from the base of a tree, a 2 m tall person measures the angle of elevation to the top of the tree to be 60°. How high is the tree?
Solution:



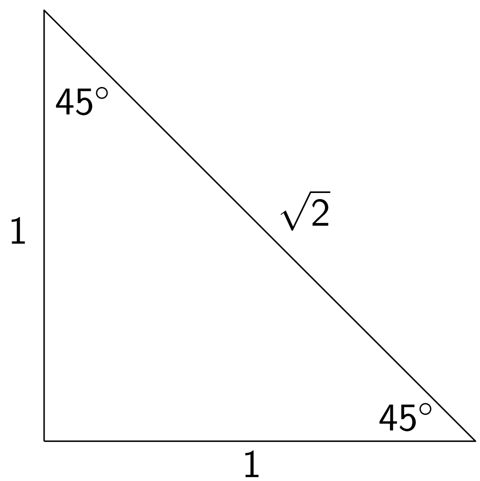
1. $y = 10 \tan 60^\circ$
2. $h = 2 + y$
3. $h = 2 + 10\sqrt{3} \approx 19.3$

The 60-90-30 Triangle



1. $\sin 60^\circ = \frac{\sqrt{3}}{2}$
2. $\cos 60^\circ = \frac{1}{2}$
3. $\tan 60^\circ = \sqrt{3}$
4. $\sin 30^\circ = \frac{1}{2}$
5. $\cos 30^\circ = \frac{\sqrt{3}}{2}$
6. $\tan 30^\circ = \frac{1}{\sqrt{3}}$

The 45-90-45 Triangle



1. $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$
2. $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$
3. $\tan 45^\circ = 1$

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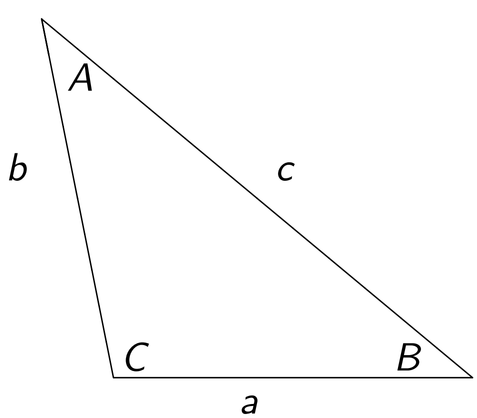
Other Trigonometric Identities

Sine Law



1. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

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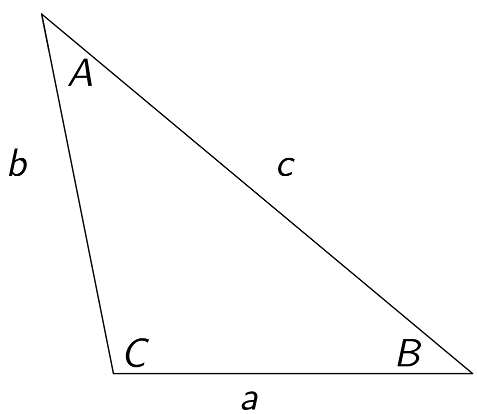
Example 2

Problem: Solve the triangle for which $a = 12, b = 10$ and $A = 70^\circ$.

Solution:

1. $\dfrac{12}{\sin 70^\circ} = \dfrac{10}{\sin B} \Leftrightarrow \sin B = \dfrac{5}{6} \sin 70^\circ \approx 0.783077$
2. $B = \sin^{-1}(0.783077) \approx 51.5^\circ$
3. $C = 180^\circ - A - B \approx 58.5^\circ$
4. $\dfrac{12}{\sin 70^\circ} = \dfrac{c}{\sin C} \Leftrightarrow c = 12 \dfrac{\sin 58.5^\circ}{\sin 70^\circ} \approx 10.9$

Cosine Law

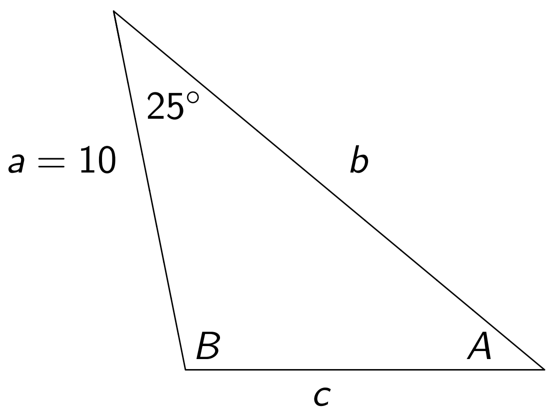


1. $a^2 = b^2 + c^2 - 2bc \cos A$
2. $b^2 = a^2 + c^2 - 2ac \cos B$
3. $c^2 = a^2 + b^2 - 2ab \cos C$

Example 3

Problem: Solve the triangle for which $a = 10$, $b = 12$ and $C = 25°$.

Solution: B must be obtuse. Why?



1. If $B = 90°$, then

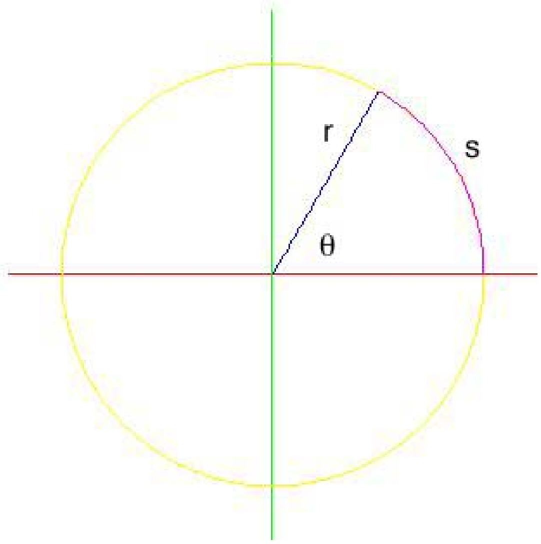
$$b = 10 \sec 25° \approx 11.03 < 12$$

2. If $B < 90°$, then $b < 11.03$.
3. So $B > 90°$.

Hence:

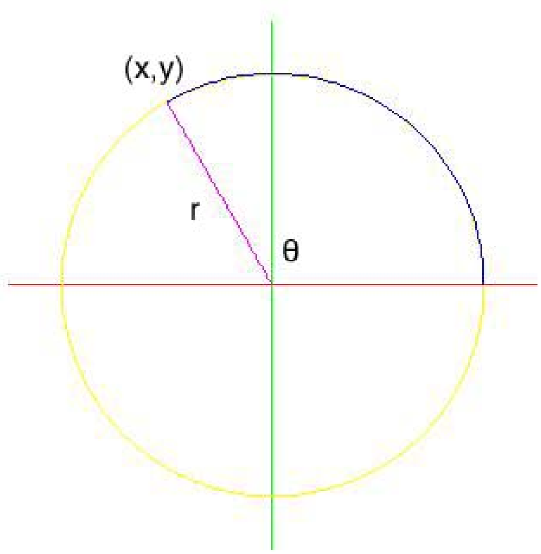
1. $c^2 = 10^2 + 12^2 - 2(10)(12)\cos 25° \approx 26.486 \Rightarrow c \approx 5.146$
2. $\dfrac{12}{\sin B} = \dfrac{5.146}{\sin 25°} \Leftrightarrow \sin B = \dfrac{12}{5.146} \sin 25° \approx 0.985507$
3. So $B = 80.2°$ or $99.8°$. Take $B = 99.8°$.
4. Finally $A = 180° - B - C \approx 55.2°$.

Radian Measure and Circles



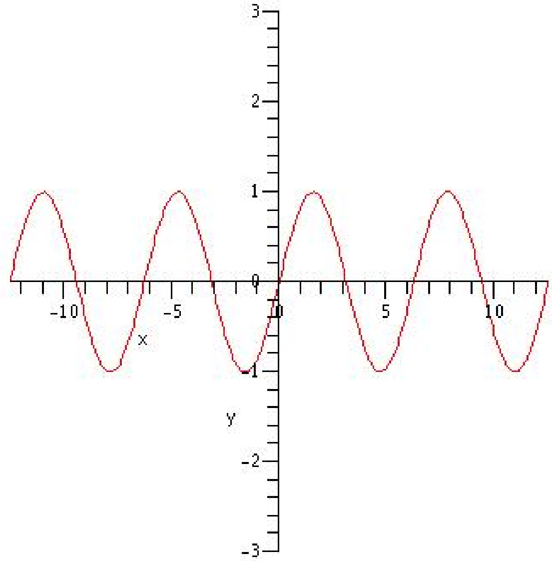
1. $\theta = \frac{s}{r}$
2. $s = r\theta$
3. π radians $= 180°$
4. Radian measure is the ratio of two lengths; so it is unit free.

The Six Trigonometric Functions; θ in radians.



1. $\sin\theta = \frac{y}{r}$
2. $\cos\theta = \frac{x}{r}$
3. $\tan\theta = \frac{y}{x}$
4. $\csc\theta = \frac{r}{y}$
5. $\sec\theta = \frac{r}{x}$
6. $\cot\theta = \frac{x}{y}$

Graph of $y = \sin x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



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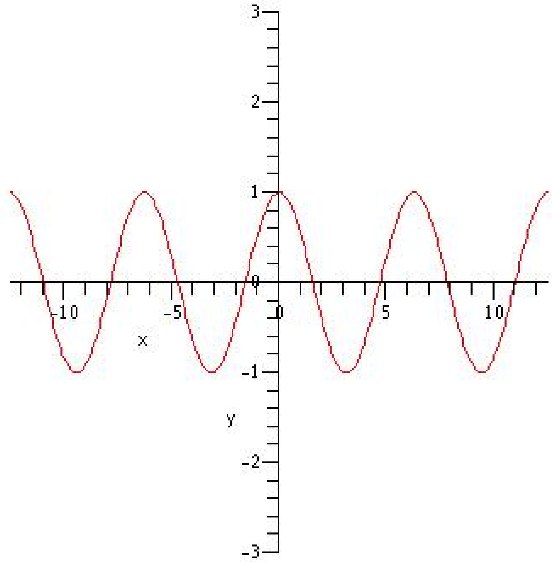
Graph of $y = \cos x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



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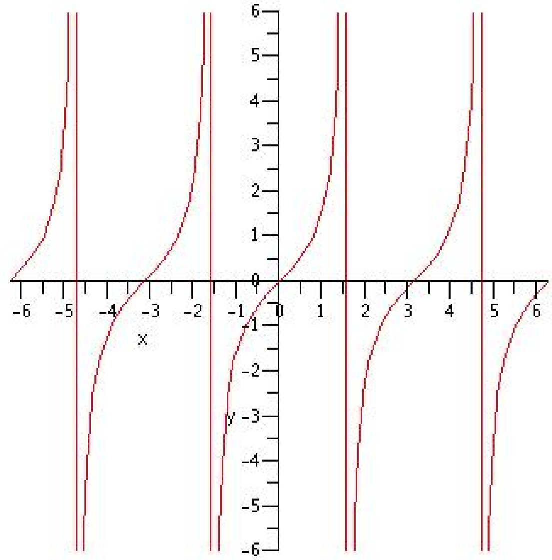
Graph of $y = \tan x$. Domain: $x \neq \dfrac{(2k+1)}{2}\pi$; Range: $\mathbb{R}$



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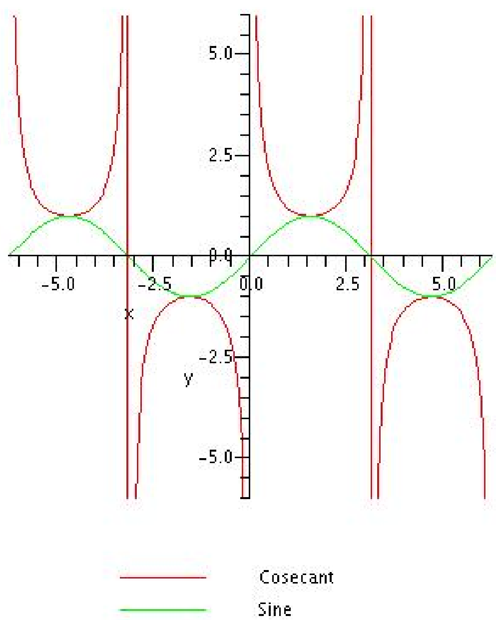
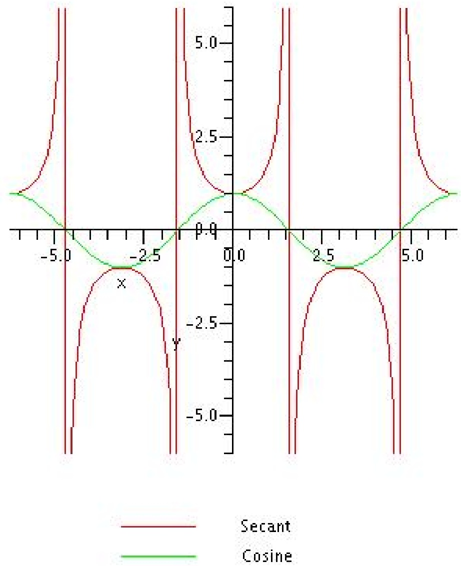
Graph of $y = \csc x$. Domain: $x \neq k\pi$; Range: $|y| \geq 1$



$y = \sec x$. Domain: $x \neq \frac{(2k+1)}{2}\pi$; Range: $|y| \geq 1$



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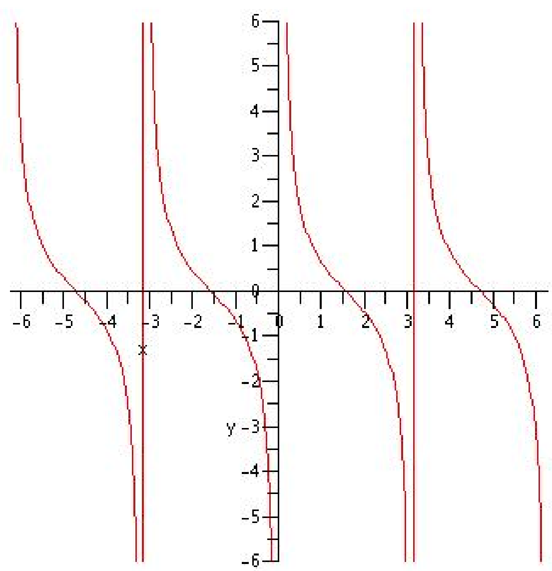
Graph of $y = \cot x$. Domain: $x \neq k\pi$; Range: $\mathbb{R}$



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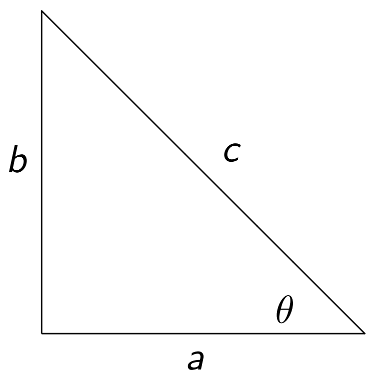
The Pythagorean Identities: assuming $a^2 + b^2 = c^2$



Then

$$\sin\theta = \frac{b}{c}; \cos\theta = \frac{a}{c}; \tan\theta = \frac{b}{a}.$$

Consequently:

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

Negative, Complementary and Supplementary Angles

Negative angles:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Supplementary angles:

$$\sin(\pi - \theta) = \sin\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$

The supplementary angle formulas follow from the negative and complementary angle formulas:

$$\begin{aligned} & \sin(\pi - \theta) \\ = & \sin\left(\pi/2 - (\theta - \pi/2)\right) \\ = & \cos(\theta - \pi/2) \\ = & \cos(\pi/2 - \theta) \\ = & \sin\theta \end{aligned}$$

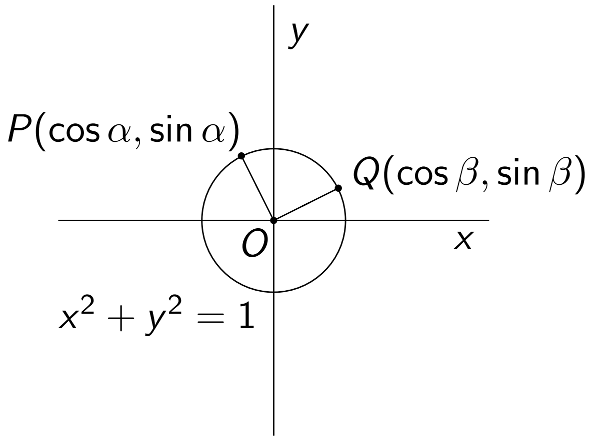
Sum and Difference Formulas

1. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
2. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
3. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
4. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

By using identities from the previous slide it can be shown that the first three above formulas follow from the fourth formula. For example:

$$\begin{aligned}\sin(\alpha+\beta) &= \cos(\pi/2-(\alpha+\beta))\\ &= \cos(\pi/2-\alpha)\cos\beta + \sin(\pi/2-\alpha)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

One Proof of $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



1. Let P have coordinates $(\cos\alpha, \sin\alpha)$.
2. Let Q have coordinates $(\cos\beta, \sin\beta)$.
3. Let $\theta = \alpha - \beta$.
4. Use the cosine law to calculate $\overline{PQ}$.

$$\begin{aligned} &\overline{PQ}^2 = \overline{OP}^2 + \overline{OQ}^2 - 2\overline{OP}\cdot\overline{OQ}\cos\theta\\ \Leftrightarrow\ &(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = 1^2+1^2-2\cos(\alpha-\beta)\\ \Leftrightarrow\ &2\cos(\alpha-\beta) = 2-2+2\cos\alpha\cos\beta+2\sin\alpha\sin\beta\\ \Leftrightarrow\ &\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

Double Angle Formulas

By taking the special cases when $\alpha = \beta = \theta$, the formulas for

$$\sin(\alpha + \beta) \text{ and } \cos(\alpha + \beta)$$

can be written as double angle formulas:

1. $\sin(2\theta) = 2 \sin\theta \cos\theta$
2. There are three formulas for $\cos(2\theta)$:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \text{or} &= 1 - 2\sin^2\theta \\ \text{or} &= 2\cos^2\theta - 1 \end{aligned}$$

The last two can be re-written as

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \text{ and } \cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

Half-Angle Formulas

Substitute $A = 2\theta$ in the last two formulas of the previous slide, and you obtain

$$\sin^2\frac{A}{2} = \frac{1 - \cos A}{2} \text{ and } \cos^2\frac{A}{2} = \frac{1 + \cos A}{2}.$$

These formulas can be used to obtain

$$\sin\frac{A}{2} \text{ and } \cos\frac{A}{2}$$

in terms of $\cos A$, but you need to know which quadrant $\frac{A}{2}$ is in to know which square root – positive or negative – to use.

Example 4

Problem: Find the exact value of both $\sin \frac{3}{4}\pi$ and $\cos \frac{\pi}{12}$.

Solutions:

1. $\sin \frac{3}{4}\pi = \sin\left(\pi - \frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

2. $$\begin{aligned} \cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} \end{aligned}$$

Example 5

Problem: Given that $\sin\theta = \frac{1}{4}$ and that $\cos\theta < 0$, find the exact values of both

$$\cos\theta \text{ and } \sin(2\theta).$$

Solutions:

1. $\cos\theta = -\sqrt{1 - \left(\frac{1}{4}\right)^2} = -\frac{\sqrt{15}}{4}$

2. $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{15}}{8}$

Example 6

Problem: Given that $\cos\theta = \frac{2}{3}$, $\sin\theta < 0$, and that $0 < \theta < 2\pi$, find the exact values of $\cos(2\theta)$, $\sin\theta$ and $\sin\left(\frac{\theta}{2}\right)$.

Solutions: θ is in the fourth quadrant. (Why?)

1. $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{2}{3}\right)^2 - 1 = -\frac{1}{9}$
2. $\sin\theta = -\sqrt{1 - \left(\frac{2}{3}\right)^2} = -\frac{\sqrt{5}}{3}$
3. $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} = \frac{1}{6} \Rightarrow \sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{6}}$,

since

$$3\pi/2 < \theta < 2\pi \Rightarrow 3\pi/4 < \theta/2 < \pi.$$

Example 7

Problem: Find all solutions x in the interval $[-\pi/2, \pi/2]$ to the equation $8\sin^2 x\cos^2 x = 1$. **Solution:**

$$\begin{aligned}
8\sin^2 x\cos^2 x = 1 \quad &\Leftrightarrow \quad 2(4\sin^2 x\cos^2 x) = 1 \\
&\Leftrightarrow \quad 2(2\sin x\cos x)^2 = 1 \\
&\Leftrightarrow \quad 2\sin^2(2x) = 1 \\
&\Leftrightarrow \quad \sin^2(2x) = \frac{1}{2} \\
&\Leftrightarrow \quad \sin(2x) = \frac{1}{\sqrt{2}} \text{ or } \sin(2x) = -\frac{1}{\sqrt{2}} \\
&\Rightarrow \quad 2x = \frac{\pi}{4}, \frac{3\pi}{4} \text{ or } 2x = -\frac{\pi}{4}, -\frac{3\pi}{4} \\
&\Rightarrow \quad x = \frac{\pi}{8}, \frac{3\pi}{8} \text{ or } x = -\frac{\pi}{8}, -\frac{3\pi}{8}
\end{aligned}$$

What is an Inverse Function?

If

1. $g(f(x)) = x$ for all x in the domain of f, and
2. $f(g(x)) = x$ for all x in the domain of g,

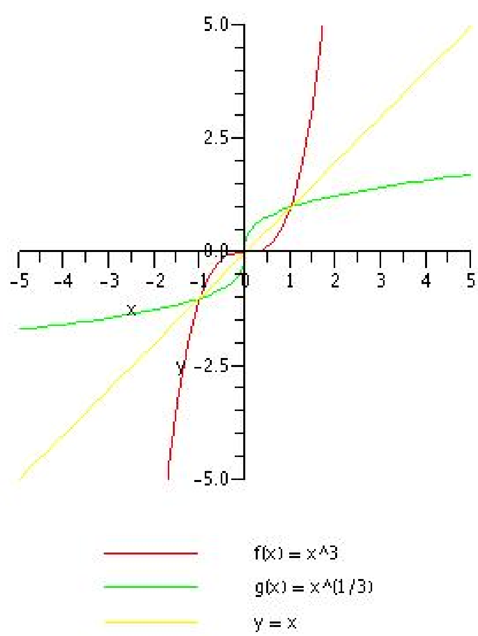
then we say f and g are inverse functions, and we write

$$f^{-1} = g \text{ and } g^{-1} = f.$$

For example, $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverse functions, since

1. $g(f(x)) = g(x^3) = (x^3)^{1/3} = x^{3/3} = x^1 = x$, and
2. $f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x^{3/3} = x^1 = x$.

The Graphs of $f(x) = x^3$ and $g(x) = x^{1/3}$.



How to Tell if a Function Has an Inverse?

The function f has an inverse if and only if it is one-to-one. This means:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Geometrically, a function f is one-to-one if any horizontal line that intersects the graph of f intersects it in exactly one point.

Examples:

1. $f(x) = 2x$ is one-to-one: $2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
2. $f(x) = x^2$ is not one-to-one: $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$.
3. $f(x) = x^3$ is one-to-one:

$$\begin{aligned} x_1^3 = x_2^3 \quad &\Rightarrow \quad x_1^3 - x_2^3 = 0 \\ &\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0 \\ &\Rightarrow \quad x_1 = x_2, \text{ since } x_i \text{ is real.} \end{aligned}$$

How To Find $f^{-1}(x)$?

If $y = f(x)$ is a one-to-one function, then the formula for $f^{-1}(x)$ can be found by interchanging x and y and solving for y. That is:

$$x = f(y) \Leftrightarrow y = f^{-1}(x).$$

For example, if $f(x) = x^3$ then

$$x = f(y) \Leftrightarrow x = y^3 \Leftrightarrow y = x^{1/3} \Leftrightarrow f^{-1}(x) = x^{1/3}.$$

Consequently:

1. (x, y) is on the graph of f if and only if (y, x) is on the graph of f^{-1}.
2. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.
3. The domain of f is the range of f^{-1}.
4. The range of f is the domain of f^{-1}.

Example 1

Suppose $f(x) = \dfrac{x+1}{x-2}$. Is f one-to-one? Yes:

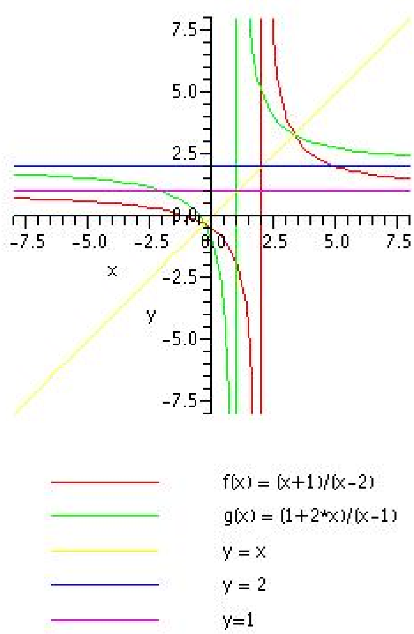
$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow \frac{x_1+1}{x_1-2} = \frac{x_2+1}{x_2-2} \\ &\Rightarrow x_1x_2 - 2x_1 + x_2 - 2 = x_1x_2 + x_1 - 2x_2 - 2 \\ &\Rightarrow 3x_2 = 3x_1 \\ &\Rightarrow x_2 = x_1 \end{aligned}$$

To find $f^{-1}(x)$:

$$x = f(y) \Leftrightarrow x = \frac{y+1}{y-2} \Leftrightarrow xy - 2x = y + 1 \Leftrightarrow y = \frac{1+2x}{x-1}.$$

Hence: $$f^{-1}(x) = \frac{1+2x}{x-1}.$$

Graphs for Example 1



Example 2

Suppose $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. Observe that f is odd: $f(-x) = -f(x)$.

$$\text{Then:} \qquad f(x_1) = f(x_2) \Rightarrow \frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}}$$

$$\text{(cross-multiplying, squaring)} \Rightarrow x_1^2(x_2^2+1) = x_2^2(x_1^2+1)$$

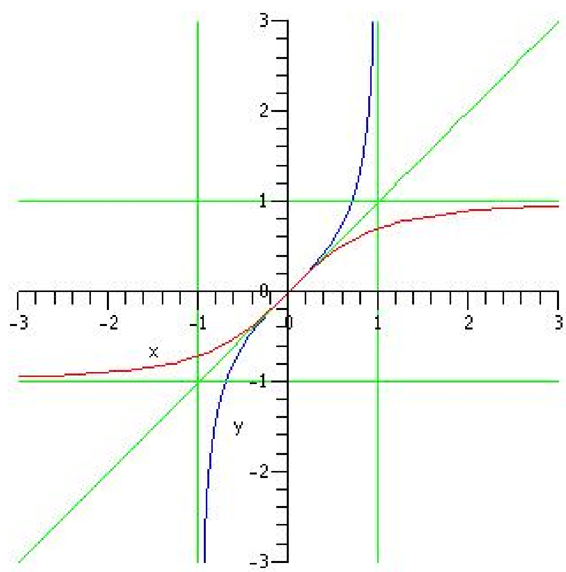
$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = x_2, \text{ since } f \text{ is odd.}$$

So f is one-to-one. To find $f^{-1}(x) : x = f(y) \Leftrightarrow x = \dfrac{y}{\sqrt{y^2+1}}$

$$\Rightarrow x^2y^2 + x^2 = y^2 \Rightarrow y^2 = \frac{x^2}{1-x^2} \Rightarrow y = \frac{x}{\sqrt{1-x^2}}, \text{ since } xy \geq 0.$$

Graphs for Example 2. Label!



How Can Trigonometric Functions have Inverses?

- Since all the trigonometric functions are periodic, they are not one-to-one. Properly speaking no trigonometric function can have an inverse.
- However the problem of finding angles with given trigonometric ratios is a common problem, which shows up in lots of applications.
- So by convention, we agree to define the inverse of trigonometric functions in particular, standard ways.
- You will find that your calculator has these standard definitions built in for $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.
- Surprisingly, there is as yet no universally agreed upon definition of $\sec^{-1} x$; so you won't find it on your calculator.

Notation for Inverse Trigonometric Functions

- f^{-1} is the common notation for the inverse of the one-to-one function f. But with respect to trig functions it can cause confusion.
- $\sin^{-1} x$ could be misread as $(\sin x)^{-1} = \csc x$. To avoid this potential problem there is an alternate notation: $\arcsin x$.
- Thus the four inverse trig functions we shall be using, namely

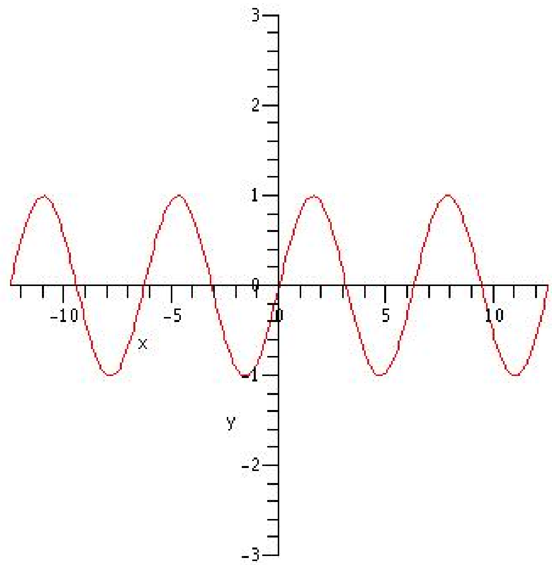
$$\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x,$$

can all be written as

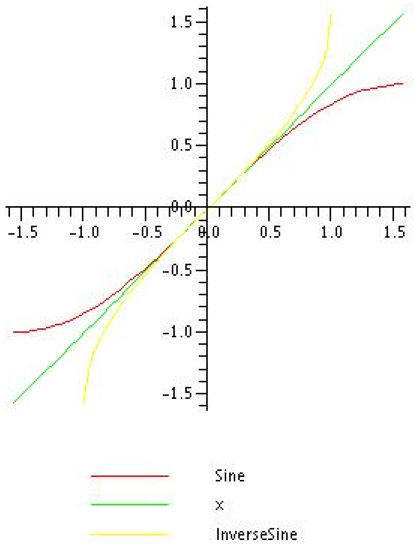
$$\arcsin x, \arccos x, \arctan x, \text{arcsec } x.$$

- Be familiar with each notation.

Graph of $y = \sin x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



Definition: $y = \sin^{-1} x \Leftrightarrow \sin y = x$ and $|y| \leq \pi/2$

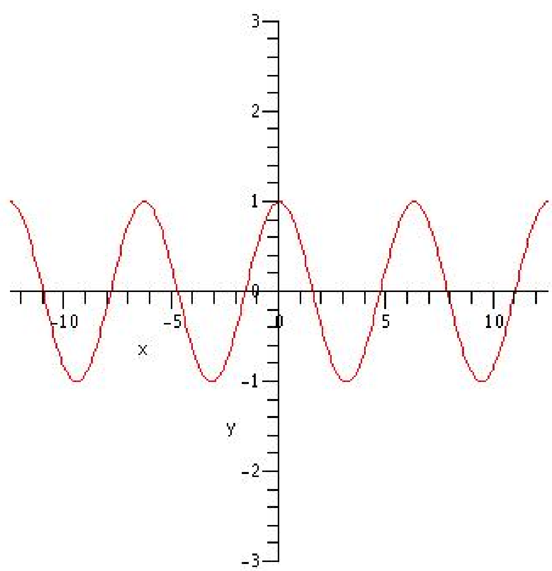


1. The domain of $y = \sin^{-1} x$ is $[-1, 1]$.
2. The range of $y = \sin^{-1} x$ is $[-\pi/2, \pi/2]$.
3. The graph of $y = \sin^{-1} x$ is always increasing.

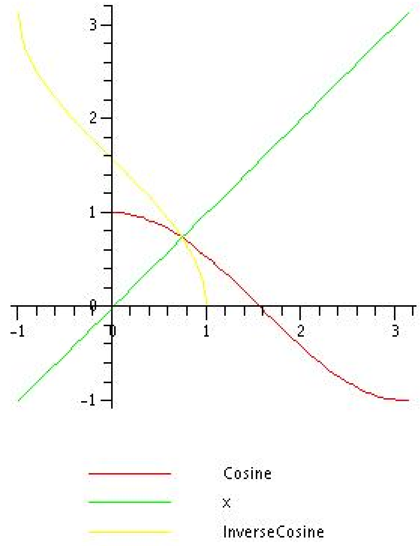
Example 3

1. $\sin^{-1} 1 = \frac{\pi}{2}$. Not 90! Even though your calculator supplies answers in terms of radians and degrees, our definition of $\sin^{-1}$ always gives values as unit-less numbers, that is, radians.
2. $\sin^{-1}(-1) = -\frac{\pi}{2}$.
3. $\sin^{-1} 0 = 0$.
4. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.
5. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Graph of $y = \cos x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



Definition: $y = \cos^{-1} x \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$

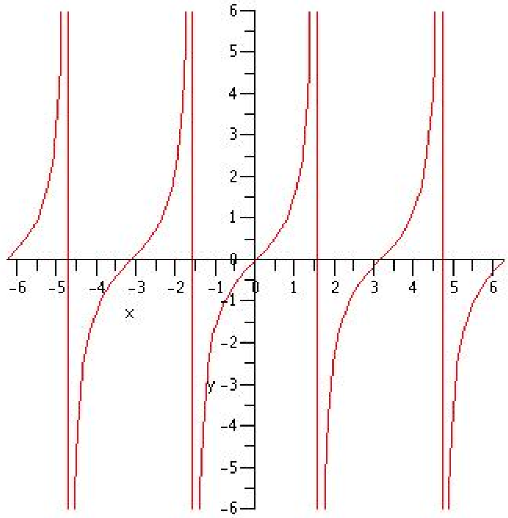


1. The domain of $y = \cos^{-1} x$ is $[-1, 1]$.
2. The range of $y = \cos^{-1} x$ is $[0, \pi]$.
3. The graph of $y = \cos^{-1} x$ is always decreasing.

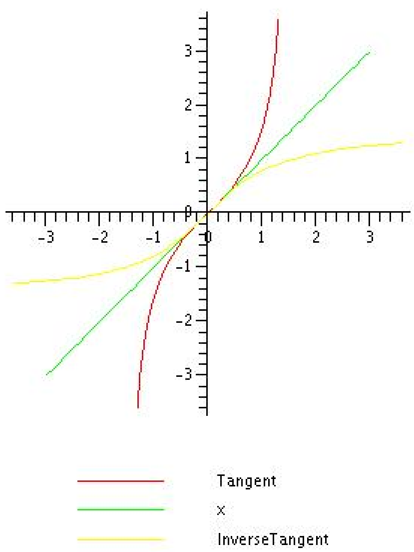
Example 4

1. $\cos^{-1} 0 = \frac{\pi}{2}$
2. $\cos^{-1} 1 = 0$
3. $\cos^{-1}(-1) = \pi$
4. $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$
5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

Graph of $y = \tan x$. Domain: $x \neq \frac{(2k+1)}{2}\pi$; Range: $\mathbb{R}$



Definition: $y = \tan^{-1} x \Leftrightarrow \tan y = x$ and $|y| < \pi/2$

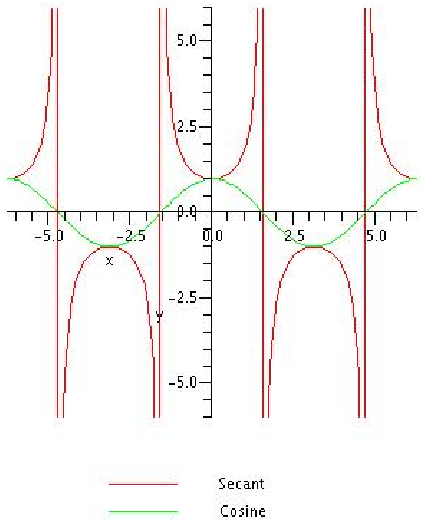


1. The domain of $y = \tan^{-1} x$ is $\mathbb{R}$.
2. The range of $y = \tan^{-1} x$ is $(-\pi/2, \pi/2)$.
3. The graph of $y = \tan^{-1} x$ is always increasing.
4. $y = \pm\frac{\pi}{2}$ are horizontal asymptotes to the graph of $y = \tan^{-1} x$.

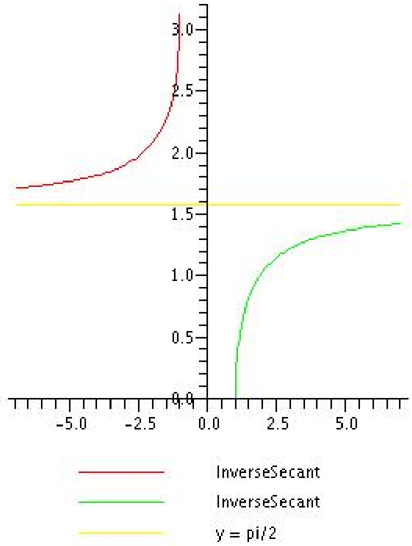
Example 5

1. $\tan^{-1} 0 = 0$
2. $\tan^{-1} 1 = \frac{\pi}{4}$
3. $\tan^{-1}(-1) = -\frac{\pi}{4}$
4. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$
5. $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$
6. $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$
7. $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$

$y = \sec x$. Domain: $x \neq \frac{(2k+1)}{2}\pi$; Range: $|y| \geq 1$



Definition: $y = \sec^{-1} x \Leftrightarrow \sec y = x$ and $0 \le y \le \pi$



1. The domain of $y = \sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$
2. The range of $y = \sec^{-1} x$ is $[0, \pi/2) \cup (\pi/2, \pi]$.
3. The graph of $y = \sec^{-1} x$ is always increasing.
4. $y = \dfrac{\pi}{2}$ is a horizontal asymptote to the graph of $y = \sec^{-1} x$.

Example 6

Warning: the definition of $\sec^{-1}$ can be quite different in different courses/books.

1. $\sec^{-1} 1 = 0$
2. $\sec^{-1} \sqrt{2} = \dfrac{\pi}{4}$
3. $\sec^{-1}(-2) = \dfrac{2\pi}{3}$
4. $\sec^{-1}(-1) = \pi$
5. $\displaystyle\lim_{x\to\infty} \sec^{-1} x = \frac{\pi}{2}$
6. $\displaystyle\lim_{x\to-\infty} \sec^{-1} x = \frac{\pi}{2}$

More About the Definition of $\sec^{-1} x$

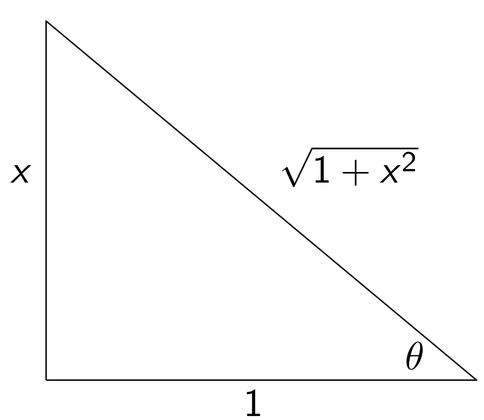
1. Our choice for the definition of $\sec^{-1} x$ has a disadvantage, and an advantage.
2. The disadvantage is that the formula for its derivative involves absolute value signs, as we shall see in Section 3.3.
3. The advantage is that $y = \sec^{-1} x$ and $y = \cos^{-1} x$ have the same range. Thus $y = \sec^{-1} x \Rightarrow x = \sec y \Rightarrow \frac{1}{x} = \cos y$

$$\Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right) \Rightarrow \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right).$$

You can use this result to evaluate $\sec^{-1} x$ with your calculator.

Example 7: Simplifying Trig Expressions with Triangles.

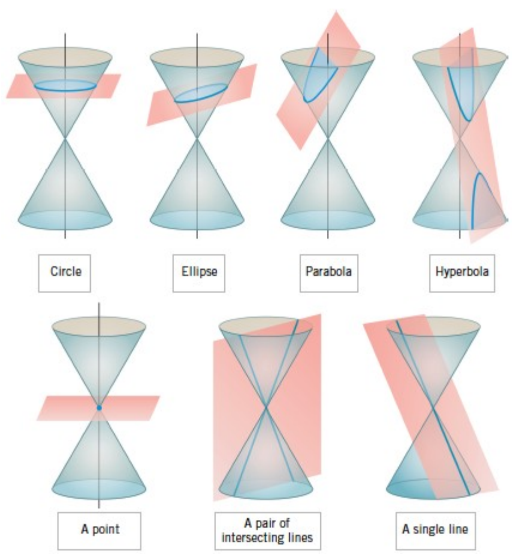
Problem: Evaluate $\sec(\tan^{-1}(x))$ in terms of x, if $x > 0$.
Solution: Let $\theta = \tan^{-1}(x)$. Then $\tan\theta = x$.



- In the triangle to the left, $x = \tan\theta$.
- The length of the hypotenuse is $\sqrt{1+x^2}$.
- Then $\sec\theta = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$.

What Is A Conic Section?

A conic section is the intersection of a plane and a cone. There are seven possibilities, three of which are 'degenerate' cases:



The four important cases are

1. The circle.
2. The ellipse.
3. The parabola.
4. The hyperbola.

The trajectories of planets or comets around the sun can be described by the above four curves, most commonly ellipses.

The Circle in the Plane

In general, a circle of radius r with centre (h, k) is defined as the set of all points (x, y) such that the distance from (x, y) to (h, k) is r. That is,

$$(x-h)^2 + (y-k)^2 = r^2.$$

For example,

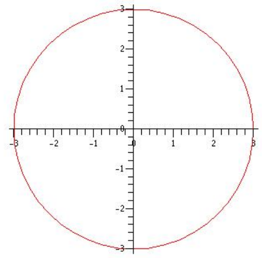


Figure: $x^2 + y^2 = 9$

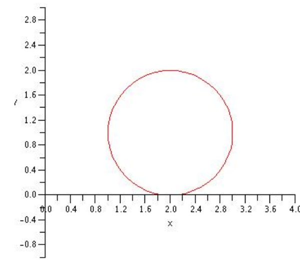


Figure: $(x-2)^2 + (y-1)^2 = 1$

Example 1

The circle with equation $(x-2)^2+(y-1)^2=1$ has equation

$$x^2-4x+4+y^2-2y+1=1 \Leftrightarrow x^2-4x+y^2-2y=-4.$$

The centre of this circle and its radius are only obvious from its equation if it is in the form

$$(x-2)^2+(y-1)^2=1.$$

To find the centre and radius of a circle if the equation is not given in factored form, you have to be able to complete the squares. That is,

$$\begin{aligned} x^2-4x+y^2-2y=-4 \quad &\Rightarrow \quad x^2-4x+4+y^2-2y+1=-4+4+1 \\ &\Rightarrow \quad (x-2)^2+(y-1)^2=1 \end{aligned}$$

Good and Bad Notation, and Some General Advice
Appendix B: Trigonometry
0.4 Inverse Functions; Inverse Trigonometric Functions
10.4 Conic Sections
The Circle
The Ellipse
The Hyperbola
The Parabola

Completing the Square in General, $a \neq 0$

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right) \\ &= a\left(x+\frac{b}{2a}\right)^2-a\left(\frac{b^2}{4a^2}-\frac{c}{a}\right) \\ &= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a} \end{aligned}$$

Either memorize this formula, or this method!

General Advice, Trigonometry and Conic Sections
MAT186H1F Lec0103 Burbulla

Example 2

Complete the square for

$$x^2 + 6x - 16.$$

We have $a = 1, b = 6, c = -16$, so

$$\begin{aligned} x^2 + 6x - 16 &= 1 \cdot \left(x + \frac{6}{2}\right)^2 - \frac{6^2 - 4 \cdot 1 \cdot (-16)}{4} \\ &= (x+3)^2 - 25 \end{aligned}$$

Or more simply:

$$x^2 + 6x - 16 = x^2 + 6x + 9 - 25 = (x+3)^2 - 25.$$

Example 3: Find the centre and the radius of the circle with equation $4x^2 + 12x + 4y^2 - 16y = -10$.

$$\begin{aligned} & 4x^2 + 12x + 4y^2 - 16y = -10 \\ \Rightarrow\; & 4(x^2 + 3x) + 4(y^2 - 4y) = -10 \\ \Rightarrow\; & \left(x^2 + 3x + \frac{9}{4} - \frac{9}{4}\right) + (y^2 - 4y + 4 - 4) = -\frac{10}{4} \\ \Rightarrow\; & \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + (y-2)^2 - 4 = -\frac{10}{4} \\ \Rightarrow\; & \left(x + \frac{3}{2}\right)^2 + (y-2)^2 = \frac{15}{4} \end{aligned}$$

So the centre is $\left(-\frac{3}{2}, 2\right)$ and the radius is $\sqrt{\frac{15}{4}}$.

The Ellipse in the Plane

In general an ellipse with foci (c_1, d_1) and (c_2, d_2) is defined as the set of all points (x, y) such that the sum of the distances from (x, y) to (c_1, d_1) and from (x, y) to (c_2, d_2) is a constant, greater than the distance between the two points (c_1, d_1) and (c_2, d_2). That is,

$$\sqrt{(x - c_1)^2 + (y - d_1)^2} + \sqrt{(x - c_2)^2 + (y - d_2)^2} = k,$$

for some constant $k > \sqrt{(c_1 - c_2)^2 + (d_1 - d_2)^2}$. The ellipse is said to be in standard position centered at the origin if the foci are on the x-axis, symmetric with respect to the origin, $(\pm c, 0)$; or if the foci are on the y-axis, symmetric with respect to the origin, $(0, \pm c)$. The graphs and equations of ellipses in standard position are shown in the next slide.

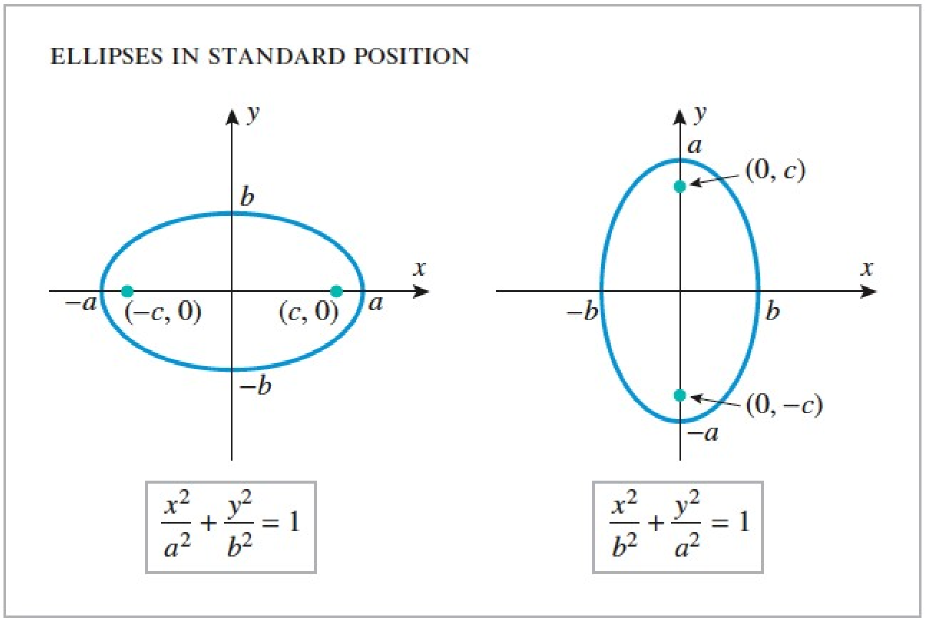


Figure: In each case, $a^2 = b^2 + c^2$

Example 4

Plot the ellipse with equation

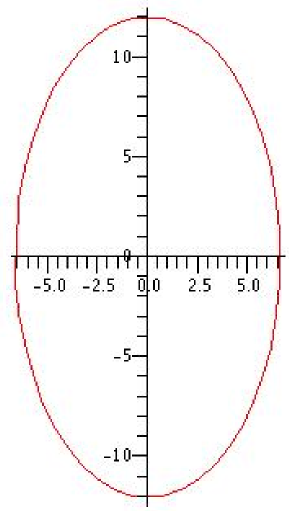
$$\frac{x^2}{44} + \frac{y^2}{144} = 1$$

and find its foci.

Solution: $a^2 = 144, b^2 = 44$, so

$$c^2 = 144 - 44 = 100.$$

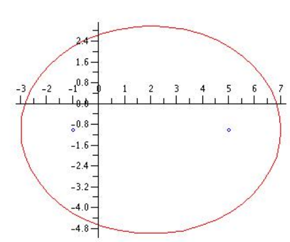
Thus the foci of the ellipse are $(0, \pm 10)$; the x-intercepts are $(\pm\sqrt{44}, 0)$; and the y-intercepts are $(0, \pm 12)$.



Example 5: By completing the squares, plot the ellipse with equation $16x^2 - 64x + 25y^2 + 50y = 311$ and find its foci.

$$\begin{aligned} & 16(x^2 - 4x) + 25(y^2 + 2y) = 311 \\ \Rightarrow\ & 16(x^2 - 4x + 4) + 25(y^2 + 2y + 1) = 311 + 64 + 25 \\ \Rightarrow\ & 16(x-2)^2 + 25(y+1)^2 = 400 \Rightarrow \frac{(x-2)^2}{25} + \frac{(y+1)^2}{16} = 1 \end{aligned}$$

The centre of this ellipse is $(2, -1)$. We have $a^2 = 25$, $b^2 = 16$, so $c^2 = 9$ and $c = \pm 3$. Thus the foci are $(2-3, -1) = (-1, 1)$ and $(2+3, -1) = (5, -1)$.



The Hyperbola in the Plane

In general a hyperbola with foci (c_1, d_1) and (c_2, d_2) is defined as the set of all points (x, y) such that the difference of the distances from (x, y) to (c_1, d_1) and from (x, y) to (c_2, d_2) is a constant, less than the distance between the two points (c_1, d_1) and (c_2, d_2). That is,

$$\sqrt{(x - c_1)^2 + (y - d_1)^2} - \sqrt{(x - c_2)^2 + (y - d_2)^2} = k,$$

for some constant $k < \sqrt{(c_1 - c_2)^2 + (d_1 - d_2)^2}$. The hyperbola is said to be in standard position centered at the origin if the foci are on the x-axis, symmetric with respect to the origin, $(\pm c, 0)$; or if the foci are on the y-axis, symmetric with respect to the origin, $(0, \pm c)$. The graphs and equations of hyperbolae in standard position are shown in the next slide.

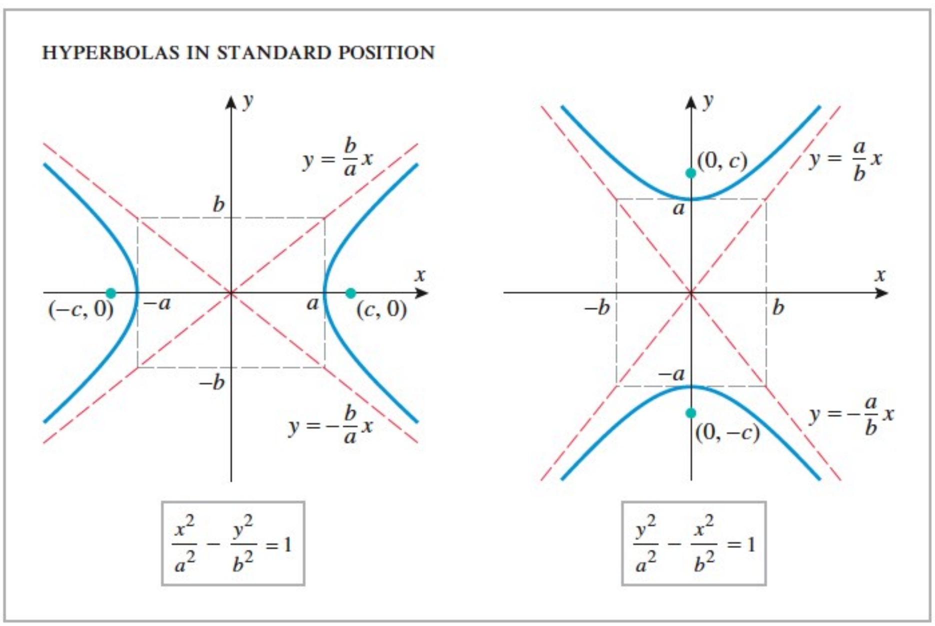


Figure: A hyperbola has two branches (blue), and two asymptotes (red). In each case, $c^2 = a^2 + b^2$.

Example 6

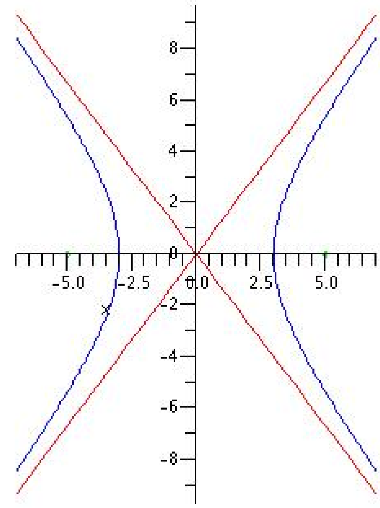
Plot the hyperbola with equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1;$$

find its foci and its asymptotes.
Solution: $a^2 = 9, b^2 = 16$, so

$$c^2 = 9 + 16 = 25.$$

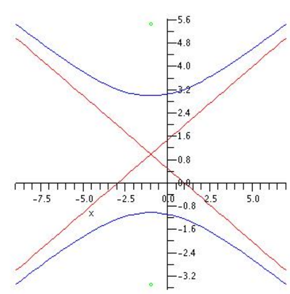
The foci of the hyperbola are $(\pm 5, 0)$; the x-intercepts are $(\pm 3, 0)$; and the asymptotes are $y = \pm\frac{4}{3}x$.



Example 7: Describe the graph of the equation $x^2 - 4y^2 + 2x + 8y = -13$.

$$\begin{aligned} & x^2 - 4y^2 + 2x + 8y = -13 \\ \Rightarrow\ & x^2 + 2x + 1 - 4y^2 + 8y - 4 = -13 + 1 - 4 \\ \Rightarrow\ & (x+1)^2 - 4(y-1)^2 = -16 \Rightarrow \frac{(y-1)^2}{4} - \frac{(x+1)^2}{16} = 1 \end{aligned}$$

The centre of this hyperbola is $(-1, 1)$. We have $a^2 = 4$, $b^2 = 16$, so $c^2 = 20$ whence $c = \pm 2\sqrt{5}$. Thus the foci are $(-1, 1 + 2\sqrt{5})$ and $(-1, 1 - 2\sqrt{5})$. The equations of the asymptotes are $y = \pm\frac{1}{2}(x+1) + 1$.



The Parabola in the Plane

In general a parabola is the set of all points in the plane that are equidistant from a fixed line, called the directrix, and a fixed point not on the line, called the focus. This definition will never be made use of in MAT186H1F, but for completeness we describe parabolas in standard position, according to this definition. A parabola is in standard position if the directrix is parallel to the x-axis, or to the the y-axis, the focus is on the coordinate axis perpendicular to the directrix, and the vertex is at the origin. The graphs and equations of parabolas in standard position are on the next slide.

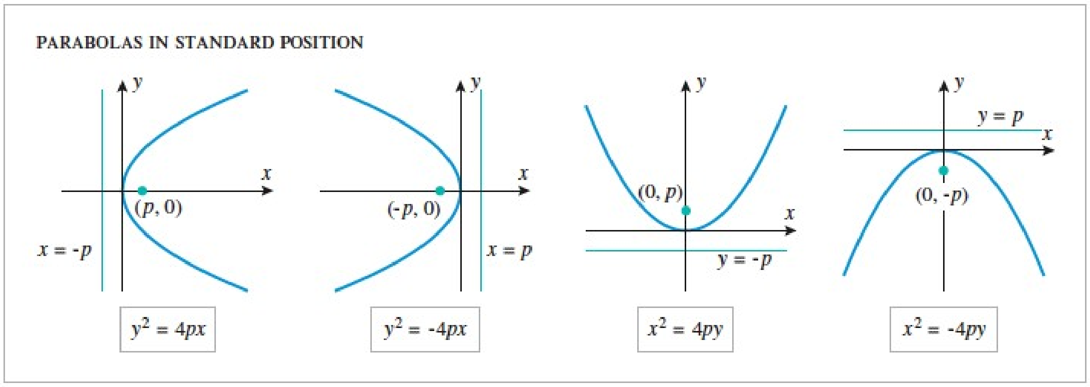


Figure: $p > 0$

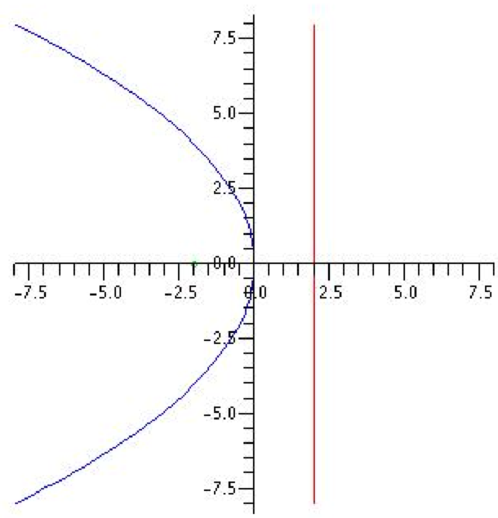
In MAT186H1F we will consider parabolas as the graphs of quadratic functions. To sketch their graphs, without calculus, we can complete the square to find the vertex of the parabola.

Example 8

Plot the parabola with equation

$$y^2 = -8x;$$

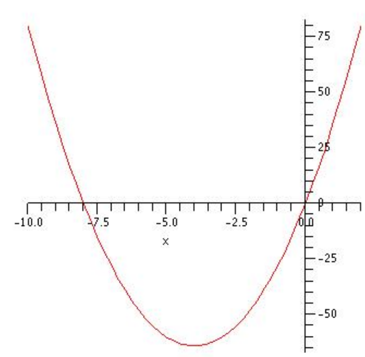
find its focus and its directrix.
Solution: $p = 2$; the focus is $(-2, 0)$ and the directrix is the line with equation $x = 2$.



Example 9: Describe the graph of the equation $y = 4x^2 + 32x$.

$$\begin{aligned} y &= 4x^2 + 32x \\ &= 4x^2 + 32x + 64 - 64 \\ &= 4(x^2 + 8x + 16) - 64 \\ &= 4(x+4)^2 - 64 \end{aligned}$$

So the vertex of this parabola is $(-4, -64)$ and the graph opens upwards. Its x-intercepts are $(-8, 0)$ and $(0, 0)$.



Example 10

The graph of the equation $xy = 1$ is a hyperbola.

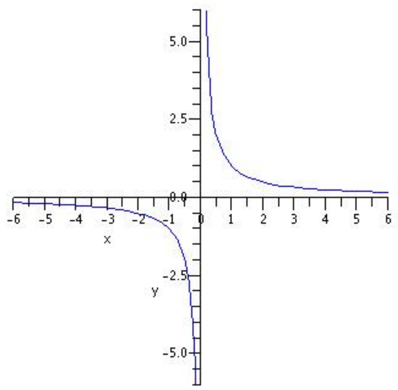


Figure: $y = 1/x$

Its asymptotes are the x-axis, and the y-axis. This hyperbola is not in standard position. Finding its foci is not an easy calculation, but we will not need to do such calculations in MAT186H1F.

However, the hyperbola with equation $xy = 1$ can be obtained by rotating the graph of the hyperbola in standard position with equation

$$\frac{x^2}{2} - \frac{y^2}{2} = 1$$

by 45°. This kind of analysis can be done using techniques in MAT188H1F Linear Algebra.