

University of Toronto
Foundational Concepts in Linear Algebra
Duration: 90 minutes

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number.

TOTAL MARKS: 100

NAME: _____

STUDENT NUMBER: _____

SIGNATURE: _____

LOCATION: (eg BA 3004) _____

MARKER'S REPORT:

| QUESTION | MARK |
|----------|------|
| Q 1 | |
| Q 2 | |
| Q 3 | |
| Q 4 | |
| Q 5 | |
| Q 6 | |
| Q 7 | |
| Q 8 | |
| Q 9 | |
| Q 10 | |
| Q 11 | |
| TOTAL | |

3. [10 marks; 5 marks for each part] A river is 45 m wide, flowing by at a rate of 2 m/sec. A rower can row a boat at a speed of 3 m/sec.

(a) If the rower rows the boat in a direction perpendicular to the bank at all times, how long will it take for the boat to cross the river? Where on the opposite bank will the boat land?

(b) At what angle should the rower aim the boat if the boat is to land on the opposite bank directly opposite the starting point? How long will it take to cross the river?

4.(a) [5 marks] How much work is required to push an 80-kg box up a ramp 10 m long, if the ramp is inclined at 13° to the horizontal?

4.(b) [5 marks] Suppose the angle between two vectors \vec{u} and \vec{a} is 40° , and that $\|\vec{u}\| = 15$, $\|\vec{a}\| = 5$. Find

- (i) the component of \vec{u} parallel to \vec{a} .
- (ii) the component of \vec{u} perpendicular to \vec{a} .

5. [10 marks] Find the **two** triangles for which $c = 2, b = 5, C = 20^\circ$.

6. [10 marks] Consider the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Find the following:

- (a) $\vec{u} + \vec{v}$
- (b) $2\vec{u} + 5\vec{v}$
- (c) $\|\vec{u}\|$
- (d) $\|\vec{v}\|$
- (e) $\vec{u} \cdot \vec{v}$
- (f) the angle between \vec{u} and \vec{v}

7. [10 marks: 5 marks for each part]

(a) Find the area of the triangle with vertices $A(1, -1, 2)$, $B(3, 3, 2)$, $C(5, 0, -4)$

(b) Use the cross product to find the sine of the angle between the two vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

What is the angle between the vectors? What is the problem with using cross products (as opposed to using dot products) to find the angle between two vectors?

8. [10 marks; 5 marks for each part]

- (a) Find parametric equations of the line passing through the point $P(8, 4, -1)$ and parallel to the line with parametric equations

$$\begin{cases} x = 2 + 5t \\ y = 9 + t \\ z = -3 + 6t \end{cases}$$

- (b) Find the point of intersection of the two lines, or show they do not intersect:

$$\begin{cases} x = 1 - 3t \\ y = 7 - 2t \\ z = 3 + t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 + s \\ y = 2 - s \\ z = 1 - s \end{cases}$$

9. [10 marks; 5 marks for each part]

- (a) Find the equation of the plane containing $A(2, 1, -1)$ and the line with parametric equations

$$\begin{cases} x = 3 \\ y = -1 - t \\ z = + 2t \end{cases}$$

- (b) find the shortest distance from the point $P(1, 0, -2)$ to the plane with equation $3x - y + 4z = 5$.

10. [10 marks] For both of the following systems of linear equations, form the augmented matrix, and solve the original system of equations by using elementary row operations to put the augmented matrix into reduced row-echelon form.

$$(a) \begin{cases} 3x_1 - x_2 = 4 \\ 2x_1 - \frac{1}{2}x_2 = 1 \end{cases} \quad (b) \begin{cases} 2x_1 + 2x_2 - 3x_3 = 1 \\ x_1 + x_3 = 5 \\ 3x_1 + 4x_2 - 7x_3 = -2 \end{cases}$$

11. [10 marks] Find conditions on the numbers a, b and c so that the given system has no solutions, a unique solution, or infinitely many solutions:

$$(a) \begin{cases} x_1 + ax_2 = 1 \\ bx_1 + 2x_2 = 5 \end{cases} \quad (b) \begin{cases} x_1 - 2x_2 + 2x_3 = a \\ -2x_1 + x_2 + x_3 = b \\ x_1 - 5x_2 + 7x_3 = c \end{cases}$$