Here are the questions from the December 2005 Final Exam in MAT188H1F, with answers. The first six questions are multiple choice.

1. If U is a subspace of $\mathbf{R}^5$ and $\dim U = 2$, then $\dim U^\perp$ is

   (a) 2

   (b) 3

   (c) 4

   (d) 5

2. $\dim\left(\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ -2 \end{bmatrix}\right\}\right) =$

   (a) 2

   (b) 3

   (c) 4

   (d) 5

3. The best approximation $Z = \begin{bmatrix} x \\ y \end{bmatrix}$ to a solution of the inconsistent system of equations

$$\begin{cases} x & & = & 14 \\ & y & = & -14 \\ 2x + & 3y & = & 0 \end{cases}$$

   is

   (a) $Z = \begin{bmatrix} 14 \\ -14 \end{bmatrix}$

   (b) $Z = \begin{bmatrix} 11 \\ -16 \end{bmatrix}$

   (c) $Z = \begin{bmatrix} -16 \\ 11 \end{bmatrix}$

   (d) $Z = \begin{bmatrix} 16 \\ -11 \end{bmatrix}$

4. What is the matrix of the transformation which is composed of a reflection in the x-axis followed by a rotation through $\frac{\pi}{2}$?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

5. The eigenspaces of the projection matrix $P_m = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$ are

(a) $E_{-1}(P_m) = \text{span}\left\{\begin{bmatrix} 1 \\ m \end{bmatrix}\right\}$ and $E_1(P_m) = \text{span}\left\{\begin{bmatrix} m \\ 1 \end{bmatrix}\right\}$

(b) $E_0(P_m) = \text{span}\left\{\begin{bmatrix} 1 \\ m \end{bmatrix}\right\}$ and $E_1(P_m) = \text{span}\left\{\begin{bmatrix} m \\ 1 \end{bmatrix}\right\}$

(c) $E_0(P_m) = \text{span}\left\{\begin{bmatrix} -m \\ 1 \end{bmatrix}\right\}$ and $E_1(P_m) = \text{span}\left\{\begin{bmatrix} 1 \\ m \end{bmatrix}\right\}$

(d) $E_1(P_m) = \text{span}\left\{\begin{bmatrix} -m \\ 1 \end{bmatrix}\right\}$ and $E_0(P_m) = \text{span}\left\{\begin{bmatrix} 1 \\ m \end{bmatrix}\right\}$

6. The equation of the plane passing through the point $(x, y, z) = (1, 0, -1)$ and perpendicular to the line $[x\ y\ z]^T = [2\ 3\ 4]^T + t[2\ 1\ 3]^T$ is

(a) $2x + 3y + 4z = -2$

(b) $2x + y + 3z = 1$

(c) $2x + 3y + 4z = 2$

(d) $2x + y + 3z = -1$

7. Suppose A is an $n \times n$ matrix such that $A^2 = O$. Explain clearly and concisely why the following six statements about A are True.

(a) $\det(A) = 0$

(b) $(A^T)^2 = O$

(c) $(I - A)^{-1} = I + A$

(d) If the system $AX = B$ is consistent, then B is in null$A$, where X and B are $n \times 1$ matrices.

(e) The only eigenvalue of A is $\lambda = 0$.

(f) If A is diagonalizable then $A = O$

7. (continued) Suppose A is an $n \times n$ matrix such that $A^2 = O$. Explain clearly and concisely why the following six statements about A are False.

   (g) $A = O$

   (h) $\text{adj}(A) = O$

   (i) A is invertible

   (j) $\text{col}A = \text{null}A$

   (k) $\text{im}A = \mathbf{R}^n$

   (l) $\dim(E_0(A)) = n$

8. Given that

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 1 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 4 & 1 \end{pmatrix} \text{ has reduced row-echelon form } R = \begin{pmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

   state the rank of A, and then find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

9. Let A be an $n \times n$ matrix; let $U = \left\{X \text{ in } \mathbf{R}^n \mid AX = A^T X\right\}$.

   (a)[5 marks] Show that U is a subspace of $\mathbf{R}^n$.

   (b)[8 marks] Let $A = \begin{bmatrix} 12 & 3 & 4 & 1 & 6 \\ 3 & 4 & 5 & 3 & 7 \\ 3 & 5 & 6 & 4 & 8 \\ 1 & 3 & 5 & 2 & 9 \\ 6 & 7 & 8 & 9 & 1 \end{bmatrix}$. Find a basis for U.

10. Let $U = \text{span}\left\{[\ 0\ \ 1\ \ 0\ \ 0\ ]^T, [\ 1\ \ -1\ \ -1\ \ 1\ ]^T, [\ 1\ \ 2\ \ -2\ \ 0\ ]^T\right\}$;

    let $X = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$. Find $\text{proj}_U(X)$ and $\text{proj}_{U^\perp}(X)$.

11. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

ANSWERS: 1.(b) 2.(b) 3.(d) 4.(a) 5.(c) 6.(d)

7.(a) $0 = \det(A^2) = (\det A)^2 \Rightarrow \det A = 0$

7.(b) $(A^T)^2 = (A^2)^T = O^T = O$.

7.(c) $(I - A)(I + A) = I - A + A - A^2 = I - O = I \Rightarrow (I - A)^{-1} = I + A$

7.(d) $AX = B \Rightarrow O = A^2X = AB \Rightarrow B$ is in null(A)

7.(e) $AX = \lambda X \Rightarrow O = A^2X = A(\lambda X) = \lambda AX = \lambda^2 X \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$

7.(f) $D = P^{-1}AP \Rightarrow D^2 = P^{-1}A^2P = O \Rightarrow \lambda_i^2 = 0 \Rightarrow \lambda_i = 0 \Rightarrow D = O$
$\Rightarrow A = PDP^{-1} = O$ OR use part (e) and start with $D = O$

7.(g),(h) and (l): can use $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as a counterexample

7.(i), (j) and (k): can use $A = O$ as a counterexample

8. rank of A is 3;

basis for row space of A is $\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix} \right\}$, or any three independent rows of A

basis for column space of A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \\ 4 \end{bmatrix} \right\}$, or any three independent columns of A

basis for the null space of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ 7 \\ -2 \\ 1 \end{bmatrix} \right\}$ 9.(a) $U = \text{null}(A - A^T)$

9.(b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ 10. $\text{proj}_U(X) = \frac{1}{6}\begin{bmatrix} 4 \\ 6 \\ -1 \\ 7 \end{bmatrix}$; $\text{proj}_{U^\perp}(X) = \frac{1}{6}\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

11. $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Here are the questions from the December 2004 Final Exam in MAT188H1F, with answers.

1. [10 marks] Let $A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}$.

   (a) [3 marks] Find the reduced row-echelon form of A^T.

   (b) [2 marks] Find the basic solutions of the homogeneous system $A^T X = O$, where X is a 4×1 matrix.

   (c) [2 marks] Let B be the matrix whose columns are the basic solutions of $A^T X = O$. Compute $B^T A$.

   (d) [3 marks] Circle the following formulas, if any, that are illustrated by the above computations.

   (i) $\left(\text{null}(A^T)\right)^\perp = \text{col}(A)$ (ii) $(\text{row}(A))^\perp = \text{null}(A)$ (iii) $\text{null}(A^T) = (\text{col}(A))^\perp$

2. [10 marks] The parts of this question are unrelated.

   (a) [4 marks] Show that the matrix $A = \begin{pmatrix} 1 & -a & b \\ a & 1 & 2 \\ -b & 0 & 1 \end{pmatrix}$ is invertible for any real values of a and b.

   (b) [6 marks] B is a 2×2 matrix such that

   $$E_2(B) = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \text{ and } E_5(B) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

   Find B.

3. [10 marks; one mark for each part] Let A be an $n \times n$ matrix. For each statement below, decide if it is equivalent to the statement

   $$A \text{ is invertible.}$$

   If it is, circle Yes at the right; if it isn't, circle No.

   (a) $\lambda = 0$ is an eigenvalue of A. Yes No

   (b) A^7 is invertible. Yes No

   (c) The rank of A is n. Yes No

   (d) AA^T is invertible. Yes No

   (e) A is a product of elementary matrices. Yes No

   (f) $\text{col}(A) = \mathbf{R}^n$ Yes No

(g) null$(A) = \mathbf{R}^n$ Yes No

(h) A is similar to the identity matrix I_n Yes No

(i) O is not in the row space of A Yes No

(j) For any vector B in $\mathbf{R}^n$, Yes No
the equation $AX = B$ has a solution.

4. [10 marks] Indicate whether each of the following statements is True or False. (Circle your choice.) It is not necessary to justify your choice, and there is no penalty for an incorrect answer.

(a) True or False: 1 is the minimum distance between the two planes with equations $x + y + z = 1$ and $x + y + z = 0$.

(b) True or False: The line of intersection of the two planes with equations $x+y+z = 3$ and $2x-3y+4z = 6$ is parallel to the vector $[\ 7 \quad -2 \quad -5\ ]^T$.

(c) True or False: If λ is an eigenvalue of the $n \times n$ matrix A, then $\lambda^2 + 4$ is an eigenvalue of the matrix $A^2 + 4I_n$.

(d) True or False: An LU-factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T$$

(e) True or False: If E and F are both $n \times n$ elementary matrices, then $(EF)^{-1} = E^{-1}F^{-1}$.

(f) True or False: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 12 \end{pmatrix} \right\}$ is an independent set in $\mathbf{R}^3$.

(g) True or False: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ -1 \\ 12 \end{pmatrix} \right\}$ is a spanning set for $\mathbf{R}^3$.

(h) True or False: Every symmetric matrix is diagonalizable.

(i) True or False: If A is an $m \times n$ matrix with rank equal to n and $A = QR$ is a QR-factorization of A, then $A^T A = R^T R$.

(j) True or False: If A is an $n \times n$ matrix such that $A^3 - A^2 + A - 2I_n = O$, then $(A^2 - A + I_n)^{-1} = 2A$.

5. [10 marks] Given that the reduced row-echelon form of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 2 & 3 & 4 & -1 \\ 2 & 2 & 6 & 4 & 2 \\ 3 & 4 & 11 & 8 & 4 \end{pmatrix} \text{ is } R = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

6. [10 marks] Solve the system of differential equations

$$\begin{array}{rcrcr} f_1'(x) & = & -f_1(x) & + & 5f_2(x) \\ f_2'(x) & = & f_1(x) & + & 3f_2(x) \end{array}$$

for f_1 and f_2 as functions of x, given the initial conditions $f_1(0) = 1$ and $f_2(0) = -1$.

7. [10 marks] Let $U = \operatorname{span}\left\{[\ 1\ \ 1\ \ 2\ \ 1\ ]^T, [\ 0\ \ 0\ \ 1\ \ 1\ ]^T, [\ 0\ \ -1\ \ 2\ \ 3\ ]^T\right\}$; let $X = \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \end{array}\right]^T$. Find $\operatorname{proj}_U(X)$.

8. [10 marks] Find the least squares approximating line for the data points

$$(1, 1), (2, 2), (3, 2), (4, 3).$$

9. [10 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

You may assume that the eigenvalues of A are $\lambda = 0$ and $\lambda = 3$.

10. [10 marks; 5 marks for each part.] The parts of this question are unrelated.

(a) Suppose $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find

$$T\begin{pmatrix} x \\ y \end{pmatrix} \text{ and } T^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) Let X and Y be any two vectors in $\mathbf{R}^n$. Prove that

X and Y are orthogonal if and only if $\|X + Y\|^2 = \|X - Y\|^2$.

ANSWERS TO DEC 2004 EXAM:

1(a) $\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 1(b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ 1(c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1(d) (i) , (iii)

2(a) $\det A = 1 + (a+b)^2 \geq 1$, so $\det A \neq 0$ 2(b) $B = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$

3(a) No (b) Yes (c) Yes (d) Yes (e) Yes (f) Yes (g) No (h) No (i) No (j) Yes

4(a) False (b) True (c) True (d) True (e) False (f) False (g) False (h) True (i) True (j) False

5. basis for row(A) is $\left\{ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$

basis for column space of A is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 4 \end{pmatrix} \right\}$

basis for null space of A is $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

6. $f_1(x) = -\frac{2}{3}e^{4x} + \frac{5}{3}e^{-2x}$; $f_2(x) = -\frac{2}{3}e^{4x} - \frac{1}{3}e^{-2x}$ 7. $\begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^T$

8. $y = 0.5 + 0.6x$ 9. $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}$

10(a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y \\ -x+y \end{pmatrix}$; $T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x+3y \end{pmatrix}$

10(b)

$$\begin{array}{rrcl} & \|X+Y\|^2 & = & \|X-Y\|^2 \\ \Leftrightarrow & (X+Y)\cdot(X+Y) & = & (X-Y)\cdot(X-Y) \\ \Leftrightarrow & X\cdot X + Y\cdot X + X\cdot Y + Y\cdot Y & = & X\cdot X - Y\cdot X - X\cdot Y + Y\cdot Y \\ \Leftrightarrow & 2X\cdot Y & = & -2X\cdot Y \\ \Leftrightarrow & X\cdot Y & = & 0 \end{array}$$

Here are the questions from the December 2003 Final Exam in MAT188H1F, with answers.

1. [10 marks] Given that the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 & -10 \\ -1 & 1 & 3 & 2 & 7 \\ 2 & -2 & 1 & 2 & 14 \\ 3 & -3 & 4 & 1 & 2 \end{pmatrix} \text{ is } R = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

2. [12 marks] Solve the system of linear ordinary differential equations

$$\begin{aligned} y_1' &= y_1 + 3y_2 \\ y_2' &= 4y_1 + 2y_2 \end{aligned}$$

for y_1 and y_2 as functions of t, given the initial conditions $y_1(0) = 1$ and $y_2(0) = 6$.

3. [10 marks] Find an orthogonal basis for the subspace of R^4 spanned by

$$\{\mathbf{w_1} = (1,1,3,2), \mathbf{w_2} = (1,-2,0,-1), \mathbf{w_3} = (0,2,1,2)\}.$$

4. [12 marks] Find the least squares line of best fit $y = a + bx$ to the four data points

$$(x_1, y_1) = (2,1), (x_2, y_2) = (3,2), (x_3, y_3) = (5,3), (x_4, y_4) = (6,4).$$

5. [12 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

6. [10 marks] Indicate whether each of the following statements is True or False. (Circle your choice.) It is not necessary to justify your choice, and there is no penalty for an incorrect answer.

   (a) True or False: $(6,4,5)$ is the point on the plane with equation $x+y-2z = 0$ that is closest to the point $(7,5,3)$.

   (b) True or False: If A is a non-zero square matrix such that $A^T = kA$, then $k = \pm 1$.

   (c) True or False: If A is a square matrix such that $A^2 = A$, then $\det A = 0$.

   (d) True or False: $\text{proj}_{(1,0,2)}(2,1,4) = (2,0,4)$

(e) True or False: $\{(1, 2, 0), (1, 1, 3), (2, 3, 3)\}$ is a spanning set for R^3.

(f) True or False: If $\mathbf{x}$ is a least squares solution to the system $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x} - \mathbf{b}$ is in the column space of A.

(g) True or False: $\{(1, 0, 1), (2, 1, 1), (0, 1, -1)\}$ is a linearly independent set of vectors in R^3.

(h) True or False: If A and B are $n \times n$ diagonal matrices, then $AB = BA$.

(i) True or False: If A and B are similar matrices, then they have the same eigenvalues and the same eigenspaces.

(j) True or False: If the nullity of the matrix A is the same as the nullity of the matrix A^T, then A must be a square matrix.

7. [34 marks] Consider the four matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}, D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(a) [2 marks] Which of the four matrices are symmetric?

(b) [2 marks] Which of the four matrices are NOT orthogonal?

(c) [2 marks] Which of the four matrices has trace 2?

(d) [4 marks] What is the determinant of each of the four matrices?

(e) [2 marks] Which of the four matrices does not have rank 3? What is its rank?

(f) [8 marks] Find the characteristic polynomial of each of the four matrices. Put your answers in factored form.

(g) [7 marks] Which of the four matrices is the standard matrix of an orthogonal projection? Find its kernel and range.

(h) [7 marks] Which of the four matrices is the standard matrix of a rotation? Find its axis of rotation, and the angle of rotation.

ANSWERS: 1. basis for row space of A : $\{(1,-1,0,0,1),(0,0,1,0,-2),(0,0,0,1,7)\}$

basis for column space of A : $\left\{\begin{pmatrix}1\\-1\\2\\3\end{pmatrix},\begin{pmatrix}2\\3\\1\\4\end{pmatrix},\begin{pmatrix}-1\\2\\2\\1\end{pmatrix}\right\}$

basis for null space of A : $\left\{\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\2\\-7\\1\end{pmatrix}\right\}$

2. $y_1 = 3e^{5t} - 2e^{-2t}$; $y_2 = 4e^{5t} + 2e^{-2t}$

3. $\left\{(1,1,3,2),\left(\frac{6}{5},-\frac{9}{5},\frac{3}{5},-\frac{3}{5}\right),\left(\frac{1}{3},0,-\frac{1}{3},\frac{1}{3}\right)\right\}$; many other correct answers

4. $y = -0.3 + 0.7x$ 5. $P = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & -1/\sqrt{3}\\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\end{pmatrix}$; $D = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$

6(a) T (b) T (c) F (d) T (e) F (f) F (g) F (h) T (i) F (j) T

7.(a) A, C (b) C (c) C (d) $\det(A) = -1; \det(B) = 1; \det(C) = 0; \det(D) = -1$

(e) rank of C is 2 (f) $\det(\lambda I - A) = (\lambda-1)^2(\lambda+1); \det(\lambda I - B) = (\lambda-1)(\lambda^2+\lambda+1)$

$\det(\lambda I - C) = \lambda(\lambda-1)^2; \det(\lambda I - D) = (\lambda+1)(\lambda^2+1)$

(g) $\text{null}(C) = \text{span}\,\{(1,0,-1)\}\,; \text{col}(C) = \text{span}\,\{(1,0,1),(0,1,0)\}$

OR could say: the range is the plane with equation $x - z = 0$ and the kernel is the line normal to the plane.

(h) axis of rotation is $\text{null}(I - B) = \text{span}\,\{(1,1,1)\}$; angle is $\pm\pi/3$, or ± 120 degrees.

Here are the questions from the December 2002 Final Exam in MAT188H1F, with answers.

1. [30 marks: 5 marks for each part] Find the following:

   (a) the inverse of $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

   (b) $\det \begin{pmatrix} 1 & 1 & 2 & 3 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 7 \\ 0 & 1 & -1 & 4 \end{pmatrix}$

   (c) the eigenvalues of the matrix $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

   (d) the coordinate vector of $\mathbf{p}(x) = -1 + 4x + 5x^2$ with respect to the basis

   $$B = \{1 + x, 1 + x^2, x + x^2\}$$

   of $\mathbf{P}_2$.

   (e) the values of a for which the matrix $\begin{pmatrix} 1 & a & 2+a \\ a & 4 & 4 \\ a & 4 & 6 \end{pmatrix}$ is not invertible.

   (f) the point on the plane with equation $x + y + z = 2$ closest to the point $(3, 2, -1)$.

2. [12 marks] Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

   $$(a + c, b + c, a + 2b + c, -a - b).$$

   Find an orthonormal basis of W. (Use the usual dot product in $\mathbf{R}^4$.)

3. [12 marks] Let W be the set of 3×3 matrices, A, satisfying the condition

   $$A^T = -A.$$

   (a) [6 marks] Show that W is a subspace of $\mathbf{M}^{3,3}$.

   (b) [6 marks] Find a basis for W and its dimension.

4. [12 marks] Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$.

5. [14 marks] Suppose A is a 3×3 invertible matrix with eigenvalues, 3, 1, and -1. Find the following:

   (a) [4 marks] the eigenvalues of A^{-1}.

   (b) [5 marks] the eigenvalues of A^T.

   (c) [5 marks] the eigenvalues of $\text{Adj}(A)$.

6. [10 marks; 2 marks for each part] Suppose $\mathbf{u}$ and $\mathbf{v}$ are two non-zero vectors in $\mathbf{R}^3$. What does each of the following conditions imply about the linear independence or dependence of the set $\{\mathbf{u}, \mathbf{v}\}$?

   (a) $\mathbf{u} = 3\mathbf{v}$

   (b) $a\mathbf{u} + b\mathbf{v} = \mathbf{0} \Rightarrow a = b = 0$

   (c) $\mathbf{u} \cdot \mathbf{v} = 0$

   (d) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$

   (e) $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ spans $\mathbf{R}^3$

7. [10 marks: 5 marks for each part.] Let $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be an orthonormal basis of an inner product space V. Prove the following:

   (a) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V,

   $$(\mathbf{u}, \mathbf{v}) = \mathbf{x} \cdot \mathbf{y},$$

   where $\mathbf{x}$ and $\mathbf{y}$ are the coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, with respect to the basis B.

   (b) If $\mathbf{x}_i$ is the coordinate vector of $\mathbf{w}_i$ with respect to the basis B, then the set $\{\mathbf{x}_1, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$ is an orthonormal basis of $\mathbf{R}^n$, with respect to the usual dot product in $\mathbf{R}^n$.

ANSWERS: 1(a) $\frac{1}{4}\begin{pmatrix} -2 & 4 & 6 \\ 1 & 2 & -1 \\ 1 & -2 & -1 \end{pmatrix}$ 1(b) 30 1(c) $\lambda = 4, \pm 3$ 1(d) $\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$

1(e) $a = \pm 2$ 1(f) $(\frac{7}{3}.\frac{4}{3}, \frac{-5}{3})$

2. $\left\{\frac{1}{\sqrt{3}}(1,0,1,-1), \frac{1}{\sqrt{3}}(-1,1,1,0), \frac{1}{\sqrt{3}}(1,1,0,1)\right\}$ is one possible answer.

3(a) Use subspace test. 3(b) $\left\{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right\}$

4. $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

5(a) reciprocals: $\frac{1}{3}, 1, -1$ 5(b) same 5(c) $-1, -3, 3$

6(a) dependent 6(b) independent 6(c) independent 6(d) dependent
6(e) independent

7(a) $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{w}_i \Rightarrow \mathbf{x} = (a_1, a_2, \ldots, a_n); \mathbf{v} = \sum_{j=1}^{n} b_j \mathbf{w}_j \Rightarrow \mathbf{y} = (b_1, b_2, \ldots, b_n)$

Then:

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) &= \left(\sum_{i=1}^{n} a_i \mathbf{w}_i, \sum_{j=1}^{n} b_j \mathbf{w}_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \left(\mathbf{w}_i, \mathbf{w}_j\right), \text{ by distribution} \\
&= \sum_{i=1}^{n} a_i b_i \left(\mathbf{w}_i, \mathbf{w}_i\right), \text{ since} \left(\mathbf{w}_i, \mathbf{w}_j\right) = 0, \text{ for } i \neq j \\
&= \sum_{i=1}^{n} a_i b_i, \text{ since } \left(\mathbf{w}_i, \mathbf{w}_i\right) = 1 \\
&= \mathbf{x} \cdot \mathbf{y}
\end{aligned}$$

7(b) Use part (a): $\mathbf{x}_i \cdot \underline{\mathrm{x}}_j = (\mathbf{w}_i, \mathbf{w}_j) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$. That is all.

Here are the questions from the December 2000 Final Exam in MAT188H1F, with answers. 403 students wrote this exam; marks ranged from 13% to 100%, with an average mark of 62.9%

1. [15 marks: avg: 12.7] Find the following:

   (a) (5 marks) $\det \begin{pmatrix} 1 & 0 & 1 & -5 \\ -1 & 3 & 1 & 0 \\ 2 & -1 & 0 & -2 \\ 0 & 1 & -1 & 4 \end{pmatrix}$

   (b) (5 marks) the standard equation of the plane passing through the three points $(1, 0, 2)$, $(1, -2, 1)$ and $(3, 0, 1)$.

   (c) (5 marks) $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1}$

2. [10 marks] avg: 7.7

   (a) [5 marks] Find the LU decomposition of the matrix $A = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & -1 \\ 1 & 0 & 5 \end{pmatrix}$.

   (Do not use any row interchanges.)

   (b) [5 marks] For s and t parameters, show that the two lines

   $$\mathbf{x}(s) = (3, -1, 2) + s(2, 1, 1) \text{ and } \mathbf{x}(t) = (1, 1, 1) + t(0, 1, -1)$$

   do not intersect.

3. [10 marks; avg: 6.7] Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $(a + b, a - b, c, a + c)$, where $a, b,$ and c are in $\mathbf{R}$. Find an *orthonormal* basis of $\mathbf{S}$, relative to the usual dot product in $\mathbf{R}^4$.

4. [20 marks: 2 marks for each part. Avg: 10.5] Indicate whether each of the following statements is true (T) or false (F), and give a brief justification for your choice:

   (a) If A and B are any 2×2 matrices, then $\det(A + B) = \det(A) + \det(B)$.

   (b) If -5 is an eigenvalue of the matrix A, then -125 is an eigenvalue of the matrix A^3.

   (c) $B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ cannot be obtained from $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ by using elementary row opertions.

   (d) If A and B are both 2×2 matrices with a common eigenvalue 1, then $A + B$ also has eigenvalue 1.

   (e) If A is a 7×7 matrix such that $A^T = -A$, then A is not invertible.

(f) If A is a square matrix such that $(A-2I)^2 = O$, then $A^{-1} = I - \frac{1}{4}A$

(g) If A is an $m \times n$ matrix, then every vector in the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is orthogonal to every vector in the row space of A, with respect to the usual dot product in $\mathbf{R}^n$.

(h) If A is an $m \times n$ matrix, $\mathbf{x}$ is in $\mathbf{R}^n$, and $\mathbf{b}$ is in $\mathbf{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in the column space of A.

(i) If A is a square matrix which is not invertible, then 0 is an eigenvalue of A.

(j) Any five 2×2 matrices must be linearly dependent.

5. [10 marks; avg: 5.2] For this question the inner product on P_2 is defined to be

$$(\mathbf{p}, \mathbf{q}) = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1).$$

   (a) (5 marks) Verify that the set $\{1, x, 3x^2 - 2\}$ is an orthogonal basis of P_2.

   (b) (5 marks) Find the coordinate vector of $\mathbf{r}(x) = x + x^2$ with respect to the basis of part (a).

6. [15 marks; avg: 9.8] Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$.

7. [10 marks; avg: 4.1] Let $\mathbf{S}$ be the subset of $\mathbf{M}^{3,3}$ consisting of all 3×3 matrices A such that the sum of the diagonal entries of A is zero.

   (a) (5 marks) Prove that $\mathbf{S}$ is a subspace of $\mathbf{M}^{3,3}$.

   (b) (5 marks) Find a basis for $\mathbf{S}$, and its dimension.

8. [10 marks; avg: 6.1] Let A be a 3×3 matrix such that

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, A\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \text{ and } A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

   Find $A\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$.

ANSWERS: 1.(a) 9 (b) $x - y + 2z = 5$ (c) $\frac{1}{4}\begin{pmatrix} -2 & 4 & 6 \\ 1 & 2 & -1 \\ 1 & -2 & -1 \end{pmatrix}$.

2.(a) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}; U = \begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \\ 0 & 0 & 16 \end{pmatrix}$.

3. $\{\frac{1}{\sqrt{3}}(1,1,0,1), \frac{1}{\sqrt{2}}(1,-1,0,0), \frac{1}{\sqrt{15}}(-1,-1,3,2)\}$

4. All are True, except for (a) and (d), which are both False.

5.(a) Show $(1,x) = 0; (1, 3x^2 - 2) = 0$ and $(x, 3x^2 - 2) = 0$ (b) $(2/3, 1, 1/3)$

6. $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; P = \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}$.

7.(b) basis $= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \right.$
$\left. \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$; $\dim\mathbf{S} = \mathbf{8}$

8. $A\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 10 \end{pmatrix}$.

Here are the questions from the December 2001 Final Exam in MAT188H1F. 451 students wrote this exam. The marks ranged from 3% to 97%, and the average was 58.1%

1. [15 marks: 5 marks for each part] Find the following:

   (a) parametric equations of the line of intersection of the two planes with equations $x + y - z = 6$ and $3x - y + 3z = 4$.

   (b) $\det \begin{pmatrix} 1 & 1 & 2 & -5 \\ -1 & 0 & 1 & 0 \\ 5 & -1 & 0 & -2 \\ 0 & 1 & -1 & 4 \end{pmatrix}$

   (c) the adjoint of $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

2.(a) [5 marks] Find the LU decomposition of the matrix $A = \begin{pmatrix} 1 & 2 & 7 \\ 1 & 5 & -1 \\ 1 & 0 & 5 \end{pmatrix}$. (Do not use any row interchanges.)

2.(b) [5 marks] Let A and B be 3×3 matrices such that $\det A = -2$ and $\det B = 4$. Find the value of $\det\left(B^2 A^T B^{-3} A^2\right)$.

3. [10 marks] Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $(a-c, a-b, c, 2a+b+c)$, where $a, b,$ and c are in $\mathbf{R}$. Find an *orthogonal* basis of $\mathbf{S}$, relative to the usual dot product in $\mathbf{R}^4$.

4. [20 marks: 2 marks for each part] Indicate whether each of the following statements is true (T) or false (F), and give a brief justification for your choice:

   (a) If E and F are any 3×3 elementary matrices, then $EF = FE$.

   (b) If 3 is an eigenvalue of the square matrix A, then 27 is an eigenvalue of the matrix A^3.

   (c) If the 6×6 matrix B is obtained from the 6×6 matrix A by replacing the third column of A with the sum of the second and fourth columns of A, then $\det B = \det A$.

   (d) If λ is an eigenvalue of the $n \times n$ matrix A, then $\lambda^2 + 1$ is an eigenvalue of $A^2 + I$, where I is the $n \times n$ identity matrix.

   (e) The set of all $n \times n$ symmetric matrices is a subspace of $\mathbf{M}^{n,n}$.

   (f) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is an orthogonal basis of $\mathbf{R}^m$, with respect to the usual dot product, and A is the $m \times m$ matrix with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ as its columns. Then the rows of $(14A)^T$ form an orthogonal basis of $\mathbf{R}^m$.

   (g) If A and B are two 2×3 matrices such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution spaces, then $A = B$.

(h) $(3, 2, -4)$ is the coordinate vector of $-3 + x - 4x^2$, relative to the ordered basis $\{x + 1, x - 1, 1 + x + x^2\}$ of P_2.

(i) The value of y in the solution of the system of equations

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & z & = & 6 \\ x & - & y & + & 3z & = & 3 \\ -x & + & y & - & 4z & = & -3 \end{array}$$

is $y = 0$.

(j) $\dim P_n = n + 1$

5. [10 marks] For which values of k does the following system of equations

$$\begin{array}{rcrcrcr} 2x & + & ky & - & z & = & 2 \\ & & y & + & kz & = & 2 \\ kx & + & y & & & = & 2 \end{array}$$

have

(a) no solutions?

(b) a unique solution?

(c) infinitely many solutions?

6. [15 marks] Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

7. [10 marks] For this question let the inner product on $\mathbf{R}^3$ be defined by

$$(\mathbf{u}, \mathbf{v}) = 2u_1v_1 + 4u_2v_2 + u_3v_3.$$

Let $\mathbf{S}$ be the subset of vectors in $\mathbf{R}^3$ which are orthogonal to $(1, -1, 2)$, with respect to the above inner product.

(a) Show that $\mathbf{S}$ is a subspace of $\mathbf{R}^3$.

(b) Find an *orthonormal* basis of $\mathbf{S}$.

8. [10 marks] Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(a) Show that for any value of θ the matrix A is orthogonal.

(b) Show that for any vector $\mathbf{v}$ in $\mathbf{R}^2$, $||A\mathbf{v}|| = ||\mathbf{v}||$, with respect to the usual dot product of $\mathbf{R}^2$.

(c) Let P be any 2×2 orthogonal matrix with determinant equal to 1. Show that $P = A$, for some value of θ.

Answers: 1.(a) $x = 5/2 - t;\ y = 7/2 + 3t;\ z = 2t$ (b) 44 (c) $\begin{pmatrix} 2 & -4 & -6 \\ -1 & -2 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

2.(a) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2/3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 3 & -8 \\ 0 & 0 & -22/3 \end{pmatrix}$ (b) -2

3. (Gram-Schmidt) $\{(1,1,0,2),(1,7,0,-4),(-12,4,11,4)\}$; one of many possible answers.

4.(a) F (b) T (c) F (d) T (e) T (f) T (g) F (h) T (i) T (j) T

5.(a) $k = -1$ (b) $k \neq 0, k \neq 1, k \neq -1$ (c) $k = 0, k = 1$

6. $C_A(x) = x(x+3)(x-3)^2$ $P = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

7.(a) $((x,y,z),(1,-1,2)) = 0 \Leftrightarrow x - 2y + z = 0$. So $\mathbf{S}$ is just a plane passing through the origin, hence it is a subspace of $\mathbf{R}^3$. (NB: you could use the subspace test and all that, if you wanted to.)

7.(b) One possible answer: $\{(1/\sqrt{3}, 0, -1/\sqrt{3}), (2/\sqrt{60}, 3/\sqrt{60}, 4/\sqrt{60})\}$

8.(a) $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Show $AA^T = I$.

8.(b) (Compute!)

$$\begin{aligned} & ||(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)|| \\ = & \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)} \\ = & \sqrt{x^2 + y^2} \\ = & ||(x,y)|| \end{aligned}$$

8.(c) Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use $PP^T = I$ and $\det P = 1$ to solve for a, b, c and d.