

Here are the questions from the MAT187H1F Final Exam of June 2006, with answers. The first six questions are Multiple Choice; each correct choice is worth 4 marks. The marks for all other questions are indicated in brackets, beside the question number. TOTAL MARKS: 100.

1. What is the fourth degree Taylor polynomial of the function $f(x) = \cos(-x)$ at $a = 0$?

(a) $1 - \frac{x^2}{2} + \frac{x^4}{4}$

(b) $1 + \frac{x^2}{2} - \frac{x^4}{4}$

(c) $1 - \frac{x^2}{2} + \frac{x^4}{24}$

(d) $1 + \frac{x^2}{2} - \frac{x^4}{24}$

2. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$?

(a) $-1 < x < 1$

(b) $-1 < x \leq 1$

(c) $-1 \leq x < 1$

(d) $-1 \leq x \leq 1$

3. Suppose a bacterial colony is growing exponentially so that its population doubles every 25 minutes. How long will it take for the population of the colony to triple in size?

(a) $25 \frac{\ln 3}{\ln 2}$ min

(b) $25 \frac{\ln 2}{\ln 3}$ min

(c) 50 min

(d) 75 min

4. A mass with $m = \frac{1}{2}$ is attached to both a spring with Hooke's constant $k = 4$ and a dashpot with damping constant $c = 3$, in appropriate units. Suppose that the mass is set in motion with initial position $x(0) = x_0$ and initial velocity $x'(0) = v_0$. Then the resulting free damped motion is:
- overdamped.
 - critically damped.
 - underdamped.
 - simple harmonic motion.
5. The area of the region inside the cardioid with equation $r = 1 + \sin \theta$ but outside the cardioid with equation $r = 1 - \sin \theta$ is equal to
- 1
 - 2
 - 4
 - 8
6. The area of the region that lies between the x -axis and the parametric curve with parametric equations $x = e^{-2t}$ and $y = \sin(3t)$, for $0 \leq t \leq \pi/3$ is given by
- $3 \int_0^{\pi/3} e^{-2t} \cos(3t) dt$
 - $-3 \int_0^{\pi/3} e^{-2t} \cos(3t) dt$
 - $-2 \int_0^{\pi/3} e^{-2t} \sin(3t) dt$
 - $2 \int_0^{\pi/3} e^{-2t} \sin(3t) dt$
7. [6 marks] Approximate $\int_0^{1/2} \frac{x^2}{(1+x^4)^{3/2}} dx$ correctly to within 10^{-4} and explain why your approximation is correct to within 10^{-4} .
8. [6 marks] The position of a particle at time t is given by $\mathbf{r} = \frac{t^2}{2} \mathbf{i} + \ln t \mathbf{j} + \sqrt{2} t \mathbf{k}$. Find the total distance travelled by the particle for $1 \leq t \leq 2$.

9. [4 marks for each part] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\sum_{n=0}^{\infty} \frac{n^3 - \sqrt{n}}{3n^4 - n + 1}$ Converges Diverges

by _____

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Converges Diverges

by _____

(c) $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ Converges Diverges

by _____

10. [13 marks] Find and classify all the critical points of $f(x, y) = 6xy^2 - 2x^3 + 3y^4$.
11. [13 marks] If x is the amount of salt dissolved in a saline solution of volume V , at time t , in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i , and r_0 is the rate at which the well-mixed solution is leaving the tank.

A 200-liter tank initially contains 100 liters of brine (i.e. saline solution) containing 20 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec, and the well-mixed brine in the tank flows out at the rate of 3 liters per sec. How much salt will the tank contain when it is full of brine?

12. [13 marks] Torricelli's Law states that

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t , $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and $g = 32 \text{ ft/sec}^2$ is the acceleration due to gravity.

A water tank is in the shape of a right circular cone with its axis vertical and its vertex at the bottom. The tank is 16 ft high and the radius of its top is 5 ft. At time $t = 0$, a plug at its vertex is removed and the tank, initially full of water, begins to drain. After 1 hour the water in the tank is 9 ft deep. When will the tank be empty?

13. [13 marks] A projectile is to be fired from the top of a 100-m cliff at a target 1 km from the base of the cliff. The projectile is fired with initial speed $v_0 = 200 \text{ m/sec}$, at an angle α to the horizontal. Find α . (Use $g = 9.8 \text{ m/sec}^2$.)

ANSWERS: 1(c) 2(b) 3(a) 4(a) 5(c) 6(d)

7. .0399925 8. $\ln 2 + \frac{3}{2}$

9(a) Diverges by Limit Comparison Test 9(b) Converges by Ratio Test

9(c) Converges by Comparison Test or by Limit Comparison Test

10. $(-1, \pm 1)$ are both minimum points; $(0, 0)$ is a saddle point

11. $200 - 20\sqrt{2} = 171.716 \dots$ kg.

12. $t = \frac{1024}{781} = 1.31114 \dots$ hr, or approximately 1 hr and 19 min after $t = 0$.

13. $\alpha = 1.3$ or 83 degrees, approximately.

Here are the questions from the MAT187H1S Final Exam of April 2006, with answers.

1. What is the fourth degree Taylor polynomial of the function $f(x) = \sin(x)$ at $a = 0$?

(a) $x + \frac{x^3}{6}$

(b) $x - \frac{x^3}{6}$

(c) $1 - \frac{x^2}{2} + \frac{x^4}{24}$

(d) $1 + \frac{x^2}{2} - \frac{x^4}{24}$

2. Suppose the half-life of a certain radioactive substance is 32 minutes. How long will it take for 75% of an initial amount of this substance to decay?

(a) 16 min

(b) 32 min

(c) 48 min

(d) 64 min

3. What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n + 5^n} (x - 1)^n$?

(a) $\frac{5}{2}$

(b) $\frac{2}{5}$

(c) $\frac{3}{2}$

(d) $\frac{2}{3}$

4. What is the slope of the tangent line to the polar graph of the polar equation $r = e^{\theta}$ at the point $(x, y) = (0, e^{(\pi/2)})$?

(a) 0

(b) -1

(c) 1

(d) $-e^{(\pi/2)}$

5. The area of the region inside the cardioid with equation $r = 2 - 2 \cos \theta$ but outside the circle with equation $r = 1$ is given by

(a) $\int_0^{\pi/3} [1 - 2 \cos \theta] d\theta$

(b) $\int_0^{\pi/3} [(2 - 2 \cos \theta)^2 - 1] d\theta$

(c) $\int_{\pi/3}^{\pi} [1 - 2 \cos \theta] d\theta$

(d) $\int_{\pi/3}^{\pi} [(2 - 2 \cos \theta)^2 - 1] d\theta$

6. What is the area of the region bounded by the curve with parametric equations $x = t^2$ and $y = t^3 - t$, for $-1 \leq t \leq 1$?

(a) $\frac{2}{15}$

(b) $\frac{4}{15}$

(c) $\frac{8}{15}$

(d) $\frac{11}{15}$

7. [6 marks] Approximate $\int_0^{1/2} \frac{1}{\sqrt{1+x^6}} dx$ correctly to within 10^{-4} , and explain why your approximation *is* correct to within 10^{-4} .

8. [6 marks] Find all positive values of a for which the initial value problem

$$y'' + ay = 0, y(0) = 0, y(1) = 0$$

has non-trivial (that is, not identically zero) solutions.

9. [4 marks each] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\sum_{n=0}^{\infty} \frac{\sin^2 n}{n^4 + 1}$ Converges Diverges

by _____

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ Converges Diverges

by _____

(c) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ Converges Diverges

by _____

10. [13 marks] Find and classify all the critical points of $f(x, y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.

11. If x is the amount of salt dissolved in a saline solution of volume V , at time t , in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i , and r_0 is the rate at which the well-mixed solution is leaving the tank.

A tank initially contains 10 liters of pure water. Saltwater containing 10 grams of salt per liter enters the tank at 1 liter per min and the (perfectly mixed) solution leaves the tank at 2 liters per min.

- (a) [2 marks] How many minutes will it take until the tank is empty?
 (b) [8 marks] Find the amount of salt (in grams) in the tank after t min.
 (c) [3 marks] What is the maximum amount of salt in the tank, at any one time?
12. [13 marks] Torricelli's Law states that

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t , $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and $g = 32 \text{ ft/sec}^2$ is the acceleration due to gravity.

The shape of a water tank is obtained by revolving the curve $y = x^{4/3}$ around the y -axis (units on the coordinate axes are in feet). A plug at the bottom is removed at 12 noon, when the water depth in the tank is 12 ft. At 1 PM the water depth is 6 ft. When will the tank be empty?

13. Water issues from the nozzle of a fire hose with speed S meters per sec.
- (a) [6 marks] Suppose the hose is held at ground level, D meters from a wall, and aimed at angle α to the horizontal. Write down parametric equations for the trajectory of the water, with x and y in terms of time t . Assume the origin $(x, y) = (0, 0)$ is chosen to be the base of the wall.
- (b) [7 marks] Show that the maximum height attainable by the water on the wall is given by

$$\frac{S^4 - g^2 D^2}{2gS^2},$$

where g is the acceleration due to gravity.

ANSWERS: 1(b) 2(d) 3(a) 4(b) 5(d) 6(c)

7. 0.499442 8. $a = \pi^2, 4\pi^2, 9\pi^2, \dots, n^2\pi^2, \dots$

9.(a) Converges by comparison test.

9.(b) Converges by comparison test, or by integral test.

9.(c) Diverges by limit comparison test.

10. $(0, 0)$ and $(-1, 1/2)$ are saddle points; $(-2, 1)$ is a maximum point.

11.(a) 10 min. 11.(b) $x = 10t - t^2$ 11.(c) 25 grams

12. 1:20 PM 13.(a) $x = -D + St \cos \alpha; y = St \sin \alpha - \frac{1}{2}gt^2$

13.(b) $x = 0 \Rightarrow y = D \tan \alpha - \frac{1}{2}g \frac{D^2}{S^2} \sec^2 \alpha$; now maximize y with respect to α .

Here are the questions from the MAT187H1F Final Exam of June 2005. 106 students wrote this exam. Marks ranged from 31% to 85%; the average was 56.5%

INSTRUCTIONS: Present your solutions to all of the following questions in the exam booklets supplied. Each question is worth 10 marks.

TOTAL MARKS: 100.

1. (5 marks for each part) Find the following:

(a) the length of the curve with parametric equations

$$x = e^t; y = \sqrt{2}t; z = e^{-t}, \text{ for } 0 \leq t \leq 1.$$

(b) the first three non-zero terms in the Maclaurin series of $f(x) = x^2\sqrt{9+x^2}$.

2. Find the area of the region inside the polar curve with polar equation $r = 2 \sin(2\theta)$ and outside the circle with polar equation $r = 1$. (It may be helpful to plot the curves first.)

3. Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - A),$$

where T is the temperature of a body placed in a surrounding medium of constant temperature A , t is time, and k is a constant.

A freshly brewed cup of tea at temperature 100 C is placed on a table in a kitchen with constant room temperature 20 C. After 3 min, the temperature of the tea is 60 C. When will the temperature of the tea be 25 C?

4. Find the critical points of $f(x, y) = x^3y - 3xy + y^2$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

5. (5 marks for each part) Find the following:

(a) the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} x^n$

(b) the approximate value of $\int_0^{1/3} x^4 \tan^{-1} x dx$ correct to within 0.0001; and explain why your approximation is correct to within 0.0001

6. Consider the curve with parametric equations

$$x = 2t^2 + 8t; y = 3t^3 - 9t.$$

Find the coordinates of all critical points on this curve, and determine if each critical point is a relative maximum point, a relative minimum point, or neither.

7. Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$

(b) (3 marks) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

(c) (4 marks) $\sum_{n=1}^{\infty} \left(-\frac{\tan^{-1} n}{2}\right)^n$

8. A 200-litre tank initially contains 100 litres of brine containing 5 kg of salt. (NB: Brine is a mixture of salt and water.) Brine containing 0.1 kg of salt per litre enters the tank at the rate of 4 litres/sec, and the well-mixed brine in the tank flows out at the rate of 3 litres/sec. How much salt will the tank contain when it is full of brine?

Recall: if x is the amount of salt in the solution at time t , then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where V is the volume of the solution in the tank at time t , c_i is the concentration of the incoming solution, r_i is the rate of flow of the incoming solution, and r_0 is the rate of outflow of the well-mixed solution.

9. A projectile is to be fired from the top of a 20-m wall at a target 800 m from the base of the wall. With what velocity¹ should the projectile be fired, if it is to hit the target in exactly 6 seconds? (The acceleration due to gravity is 9.8 m/sec².)

10. Establish the formula

$$1 - \frac{1}{3} + \frac{1}{6} - \frac{1}{10} + \frac{1}{15} - \frac{1}{21} + \dots = 4 \ln 2 - 2,$$

by integrating $\int_0^1 x \ln(1+x^2) dx$ in two different ways: with series and without series.

ANSWERS: 1(a) $e - \frac{1}{e}$ 1(b) $3x^2 + \frac{1}{6}x^4 - \frac{1}{216}x^6 + \dots$ 2. $\frac{2\pi}{3} + \sqrt{3}$

3. 12 min (from 100 C) 4. (0, 0) and $(\pm\sqrt{3}, 0)$ are saddle points; $(-1, -1)$ and $(1, 1)$ are minimum points.

5(a) $-\frac{1}{2} < x \leq \frac{1}{2}$ 5(b) 0.000228624 6. min: (10, -6); max: (-6, 6); neither: (-8, -6)

7.(a) diverges 7.(b) converges 7.(c) converges 8. $\frac{155}{8}$ kg.

9. speed: approx 136 m/sec; angle: approx 11 deg from the horizontal;

or (simply) $\mathbf{v}_0 = \left(\frac{400}{3}, \frac{391}{15}\right)$ 10. Without series, $\int_0^1 x \ln(1+x^2) dx = \ln 2 - \frac{1}{2}$;

with series, $\int_0^1 x \ln(1+x^2) dx = \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - \frac{1}{40} + \frac{1}{60} - \frac{1}{84} + \dots$

¹Recall that properly speaking, velocity is a vector.

Here are the questions from the MAT187H1F Final Exam of June 2004; 78 students wrote this exam; the marks ranged from 29% to 79%. The average mark was 51.4%

1. (10 marks; 5 marks for each part. Avg: 7.6/10) Find the general solution, y as a function of x , for each of the following differential equations:

(a) $9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 325y = 0.$

(b) $\frac{dy}{dx} = 3(y - 10)\tan x.$

2. (20 marks: 5 marks for each part. Avg: 13.3/20) Find the following:

- (a) the arc length of the curve with parametric equations

$$x = 3 \cos t, y = 3 \sin t, z = 4t$$

for $0 \leq t \leq 2\pi.$

- (b) the approximate value of $\int_0^{1/2} \frac{1 - e^{-x}}{x} dx$ correct to within $10^{-4}.$

- (c) the interval of convergence, including endpoints (if any), of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} (x + 1)^n$$

- (d) the position \mathbf{r} at time t if the acceleration at time t is $\mathbf{a} = 2\mathbf{i} + e^t \mathbf{k}$ and the initial conditions are: $\mathbf{v}_0 = \mathbf{j}$ and $\mathbf{r}_0 = \mathbf{0}.$

3. (10 marks. Avg: 8.7/10) Find the critical points of $f(x, y) = x^4 + y^4 - 16xy$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

4. (12 marks. Avg: 4.7/12) Consider the curve with parametric equations

$$x = t^2 + 4t; y = t^3 - 3t.$$

Plot this curve, labelling maximum points, minimum points and inflection points, if any.

5. (12 marks; 6 marks for each part. Avg: 2.1/12) Find the *exact* value of each of the following:

(a) $\sum_{n=0}^{\infty} \frac{n^2}{3^n}.$

(b) $\int_2^{\infty} \frac{1}{x^3 \sqrt{x^2 - 1}} dx.$

6. (16 marks. Avg: 7.5/16)

(a) (6 marks) Find the area of the region inside $r^2 = 2 \cos 2\theta$ and outside $r = 1$.

(b) (10 marks) Find the area of the region inside $r^2 = \sin 2\theta$ and $r^2 = \sqrt{3} \cos 2\theta$

7. (10 marks. Avg: 3.2/10) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=0}^{\infty} \frac{n^2 - 1}{n^{5/2} - n + 2}$.

(b) (4 marks) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

(c) (3 marks) $\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}$.

8. (10 marks. Avg: 4.2/10) A 300-litre tank initially contains 100 litres of brine containing 5 kg of salt. (NB: Brine is a mixture of salt and water.) Brine containing 0.1 kg of salt per litre enters the tank at the rate of 5 litres/sec, and the well-mixed brine in the tank flows out at the rate of 3 litres/sec. How much salt will the tank contain when it is full of brine?

ANSWERS: 1(a) $y = e^{-x/3}(c_1 \cos(6x) + c_2 \sin(6x))$ 1(b) $y = c \sec^3 x + 10$

2(a) 10π 2(b) 0.4437934 2(c) $[-3/2, -1/2]$ 2(d) $\mathbf{r} = t^2 \mathbf{i} + t \mathbf{j} + (e^t - t - 1) \mathbf{k}$

3. (0, 0) is a saddle point; both (2, 2) and (-2, -2) are minimum points.

4. min at $t = 1$; max at $t = -1$; inflection points at $t = -2, t = -2 \pm \sqrt{3}$

5(a) $3/2$ 5(b) $\pi/12 - \sqrt{3}/8$ 6(a) $\sqrt{3} - \pi/3$ 6(b) $\sqrt{3}/2 - 1/2$

7(a) diverges 7(b) converges 7(c) converges 8. approx 29.04 kg

Here are the questions from the MAT187H1F June 2003 Final Exam, with answers. 45 students wrote this exam. The marks ranged from 27% to 91% with an average of 58.5%.

1. (20 marks: each part is worth 5 marks) Find the following:

(a) the unit tangent vector to the curve with parametric equations

$$x = t^3, y = \sqrt{t}, z = \ln t$$

at $t = 1$.

(b) $\int_2^{\infty} \frac{1}{x^2 - 1} dx$.

(c) the interval of convergence, including endpoints (if any), of the power series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (x - 2)^n.$$

(d) the fifth degree Taylor polynomial of $f(x) = (1 - x^2)^{-3/2}$ about $x = 0$.

2. (10 marks; 5 marks for each part.) Find the general solution, y as a function of x , for each of the following differential equations:

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

3. (10 marks) Find the critical points of $f(x, y) = 3x - x^3 - 3xy^2$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

4. (10 marks) Find the general solution, y as a function of x , if

$$\frac{dy}{dx} + \frac{2 - 3x^2}{x^3}y = 1.$$

5. (10 marks; each part is worth 5 marks) Find the following:

(a) the exact sum (*not* a decimal approximation) of $\sum_{n=0}^{\infty} \frac{n+2}{4^n}$.

(b) the value of $\int_0^1 x^2 \cos(x^2) dx$ correct to within 10^{-4} .

6. (20 marks) For this question, consider the limaçon with polar equation $r = \sqrt{3} + \sin \theta$.

(a) (4 marks) Plot the graph of the limaçon.

(b) (6 marks) Find the Cartesian **or** polar coordinates of all the critical points on your graph. (If necessary, use your calculator to approximate the coordinates.)

- (c) (5 marks) What is the area of the region enclosed by the limaçon?
 (d) (5 marks) Show that the length of the limaçon for $0 \leq \theta \leq 2\pi$ is given by the integral

$$2 \int_{-\pi/2}^{\pi/2} \sqrt{4 + 2\sqrt{3} \sin \theta} d\theta$$

but do not try to evaluate it! Instead, describe *briefly* how you could use series to approximate it.

7. (10 marks) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 - 2n + 5}$.

(b) (4 marks) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

(c) (3 marks) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{n\pi}{n+1}\right)$.

8. (10 marks) A cannon on level ground is 400 m from the base of a cliff, which is 100 m high. At what angle to the horizontal should the cannon be aimed so that a cannon ball leaving the cannon at a speed of 80 m/sec will hit a target right on the edge (at the top) of the cliff? (Assume the acceleration due to gravity is 9.8m/sec^2 .)

ANSWERS:

1.(a) $(6/\sqrt{41}, 1/\sqrt{41}, 2/\sqrt{41})$ (b) $\ln \sqrt{3}$ (c) $1 \leq x < 3$ (d) $1 + \frac{3}{2}x^2 + \frac{15}{8}x^4$

2.(a) $y = c_1e^{-x} + c_2xe^{-x}$ (b) $y = c_1e^{-x} \cos x + c_2e^{-x} \sin x$

3. $(0, \pm 1)$ are saddle points; $(1, 0)$ is a rel max point; $(-1, 0)$ is a rel min point.

4. $y = \frac{x^3}{2} + cx^3e^{1/x^2}$ 5.(a) $\frac{28}{9}$ (b) $0.2656926 \dots$

6(b) Polar coordinates of the six critical points are:

$(r, \theta) = (\sqrt{3} + 1, \pi/2), (\sqrt{3} - 1, -\pi/2), (\sqrt{3}/2, -\pi/3), (\sqrt{3}/2, 4\pi/3),$

$\left(\frac{3\sqrt{3} + \sqrt{11}}{4}, \sin^{-1}\left(\frac{\sqrt{11} - \sqrt{3}}{4}\right)\right)$ or $\left(\frac{3\sqrt{3} + \sqrt{11}}{4}, \pi - \sin^{-1}\left(\frac{\sqrt{11} - \sqrt{3}}{4}\right)\right)$

6(c) $\frac{7}{2}\pi$ (d) Use binomial series on $\sqrt{1 + \frac{\sqrt{3}}{2} \sin \theta}$ and integrate term-by-term.

7.(a) diverges (b) converges (c) converges 8. 68.6° or 35.4°

Here are the questions from the April 2003 Final Exam in MAT 187H1S, with answers. 514 students wrote this exam. The marks ranged from 6% to 96%, with an average of 62.8%

1. (20 marks: each part is worth 5 marks) Find the following:

(a) the unit tangent vector to the curve with parametric equations

$$x = t^2, y = \sqrt{t}, z = \ln t$$

at $t = 1$.

(b) $\int_0^{\infty} \frac{1}{(1+x^2)^{3/2}} dx$.

(c) the interval of convergence, including endpoints (if any), of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n} x^n.$$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n}}{(2n+1)!} x^{2n+1}$. (HINT: what is the Maclaurin series for $\sin x$?)

2. (10 marks) Solve for A as a function of t if

$$\frac{dA}{dt} + \frac{A}{100+2t} = \frac{1}{5}$$

and $A = 20$ when $t = 0$.

3. (10 marks; 5 marks for each part.) Find the general solution, y as a function of x , for each of the following differential equations:

(a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

(b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$.

4. (10 marks; each part is worth 5 marks) Find the following:

(a) the exact sum (*not* a decimal approximation) of $\sum_{n=0}^{\infty} \frac{1}{(n+1)3^n}$.

(b) the value of $\int_0^{0.5} (1+x^2)^{2/3} dx$ correct to within 10^{-4} .

5. (10 marks) Find the critical points of $f(x, y) = 2x^2 + 8xy + y^4$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

6. (20 marks) For this question, consider the cardioid with polar equation $r = 1 + \sin \theta$.
- (4 marks) Plot the graph of the cardioid.
 - (6 marks) Find the Cartesian **or** polar coordinates of all the critical points on the graph of the cardioid.
 - (5 marks) What is the area of the region enclosed by the cardioid?
 - (5 marks) Find the length of the cardioid for $0 \leq \theta \leq 2\pi$.
7. (10 marks) Do the following infinite series converge or diverge? Justify your answer.
- (3 marks) $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$
 - (3 marks) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$
 - (4 marks) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$
8. (10 marks) A boy stands on a cliff 50 m high that overlooks a river 85 m wide. If he can throw a stone at 20 m/sec, can he throw it across the river? (Assume the acceleration due to gravity is 9.8m/sec^2 .)

ANSWERS: 1(a) $\frac{1}{\sqrt{21}}(4, 1, 2)$ 1(b) 1 1(c) $-1 < x \leq 1$ 1(d) $-\frac{1}{2}\sin(2x)$

2. $A = \frac{20}{3} + \frac{2}{15}t + \frac{400}{3\sqrt{100+2t}}$

3(a) $y = C_1e^{-2x} + C_2e^{-x}$ 3(b) $y = C_1e^{2x}\cos x + C_2e^{2x}\sin x$ 4(a) $3\ln 1.5$ 4(b) 0.5270833

5. $(0, 0)$ is a saddle point; $(4, -2)$ and $(-4, 2)$ are both minimum points.

6(b) $(x, y) = (0, 0), (0, 2), \left(\pm\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$ or $\left(\pm\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$ 6(c) $\frac{3}{2}\pi$ 6(d) 8

7(a) diverges 7(b) diverges 7(c) converges 8. No.

Regarding Question 8: the actual angle to the horizontal that the boy should throw the stone to reach the point furthest from the base of the cliff is approximately 28° ; *not* 0° or 45° .

The easiest way to solve this problem is to try and find an angle to the horizontal at which the boy should throw the stone so that it lands just on the edge of the river. But no such angle exists, since the resulting quadratic equation to be solved has *no real roots*.

Here are the questions from the April 2001 Final Exam in MAT 187H1S, with answers. 374 students wrote this exam. The marks ranged from 7% to 93%, with an average of (only) 54.1%

1. (15 marks; avg: 11.7) Find the following:

(a) (5 marks) $\int \frac{\cot(\ln x)}{x} dx$

(b) (5 marks) the length of the curve with parametric equations

$$x = 4 \cos t; y = 4 \sin t; z = 3t \text{ for } 0 \leq t \leq 2\pi.$$

(c) (5 marks) $\frac{\partial^2}{\partial x \partial y} \left(\sqrt{1 + xy^2} \right)$ at the point $(x, y) = (1, 1)$

2. (15 marks; avg: 9.2) Find the general solution to each of the following differential equations:

(a) (6 marks) $\frac{dy}{dx} = 2x(y^2 + 4)$

(b) (9 marks) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x}$

3. (15 marks; avg: 7.7) Let $f(x) = \sum_{n=0}^{\infty} (-4)^n \frac{n+1}{n^2+1} x^{2n} = 1 - 4x^2 + \frac{48}{5}x^4 - \frac{128}{5}x^6 + \frac{1280}{17}x^8 - \dots$

Find the following:

(a) (2 marks) $f^{(2)}(0)$

(b) (2 marks) the 6th degree Taylor polynomial of $f(x)$ about $x = 0$

(c) (2 marks) $\lim_{x \rightarrow 0} \frac{f(x) - 1 + 4x^2}{x^4}$

(d) (4 marks) the radius of convergence for $f(x)$

(e) (5 marks) $\int_0^{0.5} x^2 f(x) dx$ correct to within .01

4. (15 marks; avg: 10.6) Consider the curve in the x - y plane with equation $x^2 - 6x + y^2 = 0$. Find the following:

(a) (5 marks) the polar equation of the curve

(b) (5 marks) the length of the curve

(c) (5 marks) the area of the region within the curve

5. (10 marks; avg: 6.1) Find the critical points of $f(x, y) = 3x^4 - 6xy^2 - 4y^3$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

6. (10 marks; avg: 3.3) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=1}^{\infty} \frac{n+2}{n^3 + \sin n}$

(b) (3 marks) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{1+n^2}}$

(c) (4 marks) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

7. (10 marks; avg: 3.2) Find the following:

(a) (5 marks) the exact sum (*not* a decimal approximation) of $\sum_{n=1}^{\infty} \frac{n}{4^n}$.

(b) (5 marks) the first four (non-zero) terms of the Maclaurin series of $f(x) = \frac{3x+7}{x^2+2x-15}$.
What is the radius of convergence for this series?

8. (10 marks; avg: 2.5) Find $\int_0^{\infty} \frac{x}{(x+3)(x^2-3x+9)} dx$.

ANSWERS: 1(a) $-\ln|\csc(\ln x)| + c$ (b) 10π (c) $\frac{3}{4\sqrt{2}}$

2.(a) $y = 2 \tan(2x^2 + c)$ (b) $y = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{c}{x^2}$

3.(a) -8 (b) $1 - 4x^2 + \frac{48}{5}x^4 - \frac{128}{5}x^6$ (c) $\frac{48}{5}$ (d) $R = 1/2$ (e) $.0273808$

4.(a) $r = 6 \cos \theta$ (It's a circle!) (b) 6π (c) 9π

5. $(0, 0)$ is a saddle point; $(1/2, -1/2)$ is a relative minimum point

6.(a) converges (b) diverges (c) converges

7.(a) $\frac{4}{9}$ 7(b) $-\frac{7}{15} - \frac{59}{225}x - \frac{223}{3375}x^2 - \frac{1331}{50625}x^3; R = 3$

8. $\frac{2\pi}{9\sqrt{3}}$

Here are the questions from the June 2001 Final MAT 187H1F Exam; 67 students wrote this exam. The marks ranged from 25% to 81%, with an average of 58.8%

1. (15 marks: 5 marks each) Find the following:

(a) $\int xe^x dx$

(b) the length of the curve with parametric equations

$$x = \cos t; y = \sin t; z = t^{3/2}$$

for $0 \leq t \leq 1$.

(c) a unit tangent vector to the curve $\mathbf{r} = \sin^{-1}t \mathbf{i} + \ln(t + 1) \mathbf{j}$ at the point for which $t = 0$.

2. (15 marks) Find the general solution to each of the following differential equations:

(a) (5 marks) $\frac{dy}{dx} = 3y + 5$

(b) (10 marks) $\frac{dy}{dx} + \frac{xy}{x^2 + 1} = \frac{1}{x}$

3. (15 marks: 5 marks each) The following three parts are *not* related.

(a) Find the 5th degree Taylor polynomial of $f(x) = \frac{x}{1-x}$ at $x = 0$

(b) Find the interval of convergence of the power series $f(x) = \sum_{n=0}^{\infty} \frac{n}{n^2 + 1} x^n$

(c) Approximate the value of $\int_0^{1/3} \frac{\sin x}{x} dx$ correct to within 0.0001, and explain why your approximation is correct to within 0.0001

4. (15 marks) Consider the cardioid with polar equation $r = 1 - \sin \theta$.

(a) (5 marks) Plot the cardioid, and label all x and y intercepts.

(b) (5 marks) Find the length of the cardioid.

(c) (5 marks) Find the area of the region within the cardioid.

5. (10 marks) Find the critical points of $f(x, y) = 4x^3 - 6xy^2 + 3y^4$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

6. (10 marks) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=1}^{\infty} \frac{n^2 + n - 1}{n^{5/2} - n^{3/2} + 4}$

(b) (3 marks) $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$

(c) (4 marks) $\sum_{n=1}^{\infty} \frac{2^n \ln n}{5^n \sqrt{n}}$

7. (10 marks) Let $f(x) = \frac{x^2}{(1+x^2)^2}$.

(a) (6 marks) Use the binomial series expansion for $(1+x^2)^{-2}$ to find the Maclaurin series for $f(x)$ and its radius of convergence.

(b) (4 marks) What is the exact value of $\sum_{n=1}^{\infty} \frac{n(-1)^n}{3^{2n}}$?

8. (10 marks) Find $\int_1^{\infty} \frac{x+1}{x^2+x^4} dx$.

ANSWERS: 1(a) $xe^x - e^x + c$ 1(b) $\frac{1}{27}(13\sqrt{13} - 8)$ 1(c) $(1/\sqrt{2}, 1/\sqrt{2})$

2(a) $y = -5/3 + ce^{3x}$ 2(b) $y = 1 + (1+x^2)^{-1/2} \ln\left(\frac{\sqrt{1+x^2}-1}{|x|}\right) + c(1+x^2)^{-1/2}$

3(a) $P_5(x) = x + x^2 + x^3 + x^4 + x^5$ 3(b) $-1 \leq x < 1$ 3(c) 0.3312752

4(b) 8 4(c) $3\pi/2$

5. $(0, 0)$ is a saddle point; $(1/2, \pm 1/\sqrt{2})$ are minima

6(a) diverges 6(b) converges 6(c) converges

7. This is a homework problem right out of the book: #12 of section 11.13

7(a) $x^2 - 2x^4 + 3x^6 - 4x^7 + \dots$ for $|x| < 1$ 7(b) -0.09 exactly

8. $1 - \pi/4 + 1/2 \ln 2$

Here are the questions from the Final Exam in MAT187H1S of April 2002; 440 students wrote this exam. Marks ranged from 11% to 94% and the average was (only) 49.7% !!

1. (10 marks; 2 marks for each part) Indicate in the blank to the right, which one of the integral expressions A to O listed below, equals the value of the quantity described on the left:

The length of the spiral with polar equation $r = \theta$, for $0 \leq \theta \leq \pi$. _____

The distance travelled by a particle along the curve with parametric equations $x = \cos(2\theta); y = \sin(2\theta); z = \theta^2$, for $0 \leq \theta \leq \pi$, where θ represents time. _____

The length around one petal of the 3-leaved rose with polar equation $r = \sin(3\theta)$ _____

The area within the cardioid with polar equation $r = 1 - \cos \theta$ _____

The area of the region inside the curve with polar equation $r = 2 + \cos \theta$, but outside the circle with polar equation $r = 5 \cos \theta$. _____

Pick your answers from:

A. $\int_0^\pi \sqrt{1 + \theta} d\theta$	B. $2 \int_0^\pi \sqrt{1 + \theta} d\theta$	C. $\int_0^\pi \sqrt{1 + \theta^2} d\theta$
D. $2 \int_0^\pi \sqrt{1 + \theta^2} d\theta$	E. $\int_0^\pi (1 - \cos \theta)^2 d\theta$	F. $2 \int_0^\pi (1 - \cos \theta)^2 d\theta$
G. $2 \int_0^\pi (1 - \cos(3\theta))^2 d\theta$	H. $\int_0^\pi (1 - \cos(3\theta))^2 d\theta$	I. $\int_0^\pi (1 + 8 \cos^2(3\theta))^2 d\theta$
J. $\int_0^{\pi/3} (1 + 8 \cos^2(3\theta))^2 d\theta$	K. $\int_0^\pi \sqrt{1 + 8 \cos^2(3\theta)} d\theta$	L. $\int_0^{\pi/3} \sqrt{1 + 8 \cos^2(3\theta)} d\theta$

M. $\int_{\pi/3}^\pi (2 + \cos \theta)^2 d\theta - \int_{\pi/3}^{\pi/2} 25 \cos^2 \theta d\theta$

N. $\int_{\pi/3}^\pi ((2 + \cos \theta)^2 - 25 \cos^2 \theta) d\theta$

O. $\int_{\pi/3}^{\pi/2} ((2 + \cos \theta)^2 - 25 \cos^2 \theta) d\theta$

2. (10 marks) Find the exact sum — not a decimal approximation — of each of the following infinite series. Put your answer in the blank to the right.

(a) (2 marks) $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ _____

(b) (3 marks) $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ _____

(c) (5 marks) $\sum_{n=0}^{\infty} \frac{n+1}{5^n}$ _____

3. (15 marks) Find the critical points of

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$$

and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

4. (10 marks) Plot the curve in the xy -plane with parametric equations

$$x = t^2 ; y = t^3 - 3t, \text{ for } -2 \leq t \leq 2.$$

Be sure to label any maximum or minimum points, and to indicate when the graph is concave up and when it is concave down.

5. (15 marks)

- (a) (5 marks) Solve for y as a function of x , if

$$\frac{dy}{dx} + y \tan x = \cos^2 x, \text{ and } y = 5 \text{ when } x = 0.$$

- (b) (10 marks) In 1992, ten castaways were stranded on an island. After finding the fresh water, the mango groves, and the man-eating tigers, they settled in for a long stay. Suppose the population, P , of castaways at time t , satisfies the differential equation

$$\frac{dP}{dt} = \frac{1}{1000}P(100 - P),$$

where t is measured in years since 1992.

- (i) (5 marks) How many castaways are on the island now, in the year 2002?
 (ii) (5 marks) Sketch a graph of P , for $t \geq 0$. What will happen to the population of castaways as more and more time passes?
6. (15 marks; 5 marks for each part)

- (a) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}x^n}{\sqrt{n}}$$

Don't forget to check convergence of the series at the endpoints of the interval!

(b) Write down the first four nonzero terms of the Maclaurin series of $\tan^{-1}x$.
 (Hint: what is $\int \frac{1}{x^2+1} dx$?)

(c) What is the maximum possible error if the fifth degree Taylor Polynomial of $\tan^{-1}x$ about $x = 0$ is used to approximate $\tan^{-1}x$, for $0 \leq x \leq \frac{1}{2}$?

7. (10 marks) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=1}^{\infty} \frac{n^2 + \ln n}{n^4 - 2n + 3}$

(b) (4 marks) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

(c) (3 marks) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. (15 marks)

(a) (10 marks) At what angle to the horizontal should a baseball with initial speed 30 m/sec be thrown from the top of a 100 m high cliff, if the baseball is to hit a target (on the flat ground below) exactly 25 m from the base of the cliff? (Assume the acceleration due to gravity is 9.8m/sec^2 ; ignore air resistance and any other forces.)

(b) (5 marks) Find $\int_0^{\infty} \frac{1}{\sqrt{e^{ax}-1}} dx$, for $a > 0$.

ANSWERS: 1. C D L E M 2(a) e^{-2} 2(b) $3/2$ 2(c) $25/16$

3. $(0, 0)$ is a minimum point; $(-5/3, 0)$ is a maximum point;
 $(-1, \pm 2)$ are saddle points

4. $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$; $\frac{d^2y}{dx^2} = \frac{3t^2 + 3}{4t^3}$

Critical points: vertical tangent at $(0, 0)$; min at $(1, -2)$; max at $(1, 2)$

Graph is concave up for $t > 0$; concave down for $t < 0$.

x -intercept is $(3, 0)$ at $t = \pm\sqrt{3}$

5(a) $y = \cos x \sin x + 5 \cos x$ 5(b)(i) 23 5(b)(ii) logistic curve

6(a) $-1/2 < x \leq 1/2$ 6(b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$ 6(c) 0.001116

7(a) converges 7(b) converges 7(c) diverges

8(a) -70.3 or 84.4 degrees to the horizontal; 8(b) π/a

Here are the questions from the June, 2002 final MAT187H1F exam. 85 students wrote this exam; the marks ranged from 7% to 89%, and the average was 57.4%

1. (30 marks: 5 marks for each part) Find the following:

(a) the length of the logarithmic spiral $r = e^\theta$ for $\pi \leq \theta \leq 2\pi$.

(b) the speed of a particle at time $t = 2$ if its position at time t is given by the parametric equations $x = t^2 + 1$; $y = t^3 - 2t$; $z = 1 - 4t$.

(c) the exact sum of the series $\sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^{n+1}$

(d) the 5th degree Taylor polynomial of $f(x) = \frac{x}{(1+x^2)^{1/3}}$ at $x = 0$

(e) the interval of convergence of the power series $f(x) = \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n+1} x^n$

(f) $\int_0^{\infty} \frac{1}{\sqrt{e^{2x} + 1}} dx$

2. (15 marks) Find the general solution, y as a function of x , to each of the following differential equations:

(a) (5 marks) $\frac{dy}{dx} = x^2 y^2 + 2y^2$

(b) (10 marks) $\frac{dy}{dx} + \frac{y}{x+1} = \frac{1}{x^2 + x - 2}$

3. (13 marks) Find the critical points of $f(x, y) = x^4 + 3xy^2 + y^2$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

4. (10 marks) Plot the polar graphs of the two polar curves with equations

$$r_1 = \sqrt{3} \sin 2\theta \text{ and } r_2 = \cos 2\theta;$$

and find the area of the common region inside both curves.

5. (10 marks) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=0}^{\infty} \frac{n^2 + n - 1}{n^3 + n^2 + 7}$

(b) (3 marks) $\sum_{n=1}^{\infty} \left(\frac{1}{\tan^{-1} n}\right)^n$

(c) (4 marks) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

6. (12 marks; 6 marks for each part) Find the following:

(a) the exact sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{3^{n-1}}$

(b) the maximum possible error if $\sum_{n=1}^5 \frac{n}{e^n}$ is used to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$.

7. (10 marks) A bomb is dropped (initial speed zero) from a helicopter hovering at a height of 800 m. A projectile is fired from a gun located on the ground 800 m west of the point directly beneath the helicopter. The projectile is supposed to intercept the bomb at a height of exactly 400 m. If the projectile is fired 1 sec after the bomb is dropped, what should be the projectile's initial speed and angle of inclination to the horizontal? (Assume the acceleration due to gravity is 9.8m/sec^2 , and ignore air resistance and any other forces.)

ANSWERS: 1.(a) $\sqrt{2}(e^{2\pi} - e^\pi)$ (b) $2\sqrt{33}$ (c) $1/20$ (d) $x - \frac{1}{3}x^3 + \frac{2}{9}x^5$
(e) $-1 \leq x < 1$ (f) $\ln(\sqrt{2} + 1)$

2.(a) $y = -\frac{1}{\frac{x^3}{3} + 2x + c}$ (b) $y = \frac{1}{3} \frac{1}{x+1} (\ln|x+2| + 2\ln|x-1| + c)$

3. $\left(-\frac{1}{3}, \pm\frac{2}{9}\right)$ are both saddle points; $(0, 0)$ is a minimum point.

4. $\text{area} = \frac{5}{24}\pi - \frac{1}{4}\sqrt{3}$ 5.(a) diverges (b) converges (c) converges

6.(a) $\frac{9}{2}$ (b) $\frac{6}{e^5} = 0.040427681\dots$

7. angle: 41.8 degrees; speed: 133.65 m/sec