Here are the questions from the Dec 2005 MAT186H1F Final Exam, with answers. Note: questions 1 through 6 are multiple choice.

1. What is the equation of the tangent line to the graph of $f(x) = \sqrt{x+2}$ at the point $(x, y) = (-1, 1)$?

   (a) $y = x + 3$

   (b) $2y = x + 3$

   (c) $y = 2x + 3$

   (d) $2y = x + 1$

2. The area of the region bounded by the two curves $f(x) = x^3$ and $g(x) = 2x^3 + 2x^2 - 3x$ is given by

   (a) $\int_{-3}^{1} (f(x) - g(x))\, dx$

   (b) $\int_{-3}^{1} (g(x) - f(x))\, dx$

   (c) $\int_{-3}^{0} (f(x) - g(x))\, dx + \int_{0}^{1} (g(x) - f(x))\, dx$

   (d) $\int_{-3}^{0} (g(x) - f(x))\, dx + \int_{0}^{1} (f(x) - g(x))\, dx$

3. Let $f(x) = x^3 + x - 1$. If Newton's method is used to approximate a solution to the equation $f(x) = 0$, beginning with $x_0 = 0$, then the value of x_2 is

   (a) 0

   (b) 1

   (c) $\frac{3}{4}$

   (d) $\frac{59}{86}$

4. Suppose $F(x) = \int_{-20}^{x^3} f(t)\, dt$, where $f(t)$ is continuous for all values of t.

   Then $F'(2) =$

   (a) $12f(8)$

   (b) $12f(2)$

   (c) $f(2)$

   (d) $f(8)$

5. $\int_{\pi/2}^{\pi} \sin x \cos^3 x \, dx =$

(a) $\int_0^{-1} u^3 \, du$

(b) $\int_{-1}^{0} u^3 \, du$

(c) $\int_{\pi/2}^{\pi} u^3 \, du$

(d) $\int_{\pi}^{\pi/2} u^3 \, du$

6. $\lim_{x \to 0} \dfrac{x - \tan^{-1} x}{x^3} =$

(a) 0

(b) $-\dfrac{1}{3}$

(c) $\dfrac{1}{3}$

(d) ∞

7. Suppose f and g are continuous functions such that

$$\int_1^9 f(x)\,dx = 6, \int_1^9 g(x)\,dx = -4 \text{ and } \int_1^{16} f(x)\,dx = 2.$$

Find the values of the following. (Put your answers on the lines at the right of each question.)

(a) $\int_1^9 (3f(x) + 2g(x))\,dx$ ANS: ____________

(b) the average value of f on $[1, 9]$ ANS: ____________

(c) $\int_9^{16} f(x)\,dx$ ANS: ____________

(d) $\int_1^9 f(x)\,dt$ ANS: ____________

(e) $\int_1^3 x f(x^2)\,dx$ ANS: ____________

(f) $\lim_{n \to \infty} \sum_{i=1}^{n} f\left(\dfrac{n + 15i}{n}\right) \dfrac{15}{n}$ ANS: ____________

8. Let $f(x) = e^{-e^{x-1}}$. Sketch the graph of $y = f(x)$, labelling all critical points, inflection points and asymptotes, if any.

9. Suppose the velocity of a particle at time t is given by $v = t^2 - 9$, for $0 \le t \le 4$. Find the following:

(a) [3 marks] the average acceleration of the particle for $0 \le t \le 4$.

(b) [4 marks] the average velocity of the particle for $0 \le t \le 4$.

(c) [6 marks] the average speed of the particle for $0 \le t \le 4$.

10. A tank is in the shape of a cone with its vertex at the bottom. The radius at the top of the tank is r metres; the vertical height of the tank is h metres. It is full of liquid with density ρ, in units of kilograms per cubic metre. Find the work done in pumping out the top $\frac{1}{2}h$ metres of the liquid to the top of the tank.

11.(a) [7 marks] Find the volume of the solid generated by rotating around the x-axis the region in the plane bounded by the curves

$$y = \sin x + \cos x; y = 0; x = 0; x = \frac{\pi}{4}.$$

11.(b) [6 marks] Set up an integral that gives the volume of the solid generated by rotating around the y-axis the region inside the ellipse with equation $4y^2 + (x-2)^2 = 1$. (Four BONUS marks if you can evaluate it.)

12. Let $f(x) = \int_1^x \sqrt{t^5 + t^2 - 1}\, dt$, for $1 \le x \le 2$.

(a) [4 marks] Show that the arc length differential, ds, for the function $f(x)$ is given by

$$ds = \sqrt{x^5 + x^2}\, dx.$$

(b) [9 marks] Find the area of the surface of revolution generated by revolving the curve $y = f(x)$, for $1 \le x \le 2$, around the y-axis. (You may use the result from part (a).)

ANSWERS: 1.(b) 2.(d) 3.(c) 4.(a) 5.(b) 6.(c)

7.(a) 10 7.(b) 3/4 7.(c) -4 7.(d) $8f(x)$ 7.(e) 3 7.(f) 2

8. graph always decreasing; inflection point at $(1, e^{-1})$;
horizontal asymptotes: $y = 0$ and $y = 1$; no other asymptotes

9.(a) 4 9.(b) $-\frac{11}{3}$ 9.(c) $\frac{16}{3}$ 10. $\frac{11}{192}\rho\pi g r^2 h^2$

11.(a) $\frac{\pi^2}{4} + \frac{\pi}{2}$ 11.(b) $\int_1^3 2\pi x\sqrt{1-(x-2)^2}\, dx = 2\pi^2$

12.(a) $f'(x) = \sqrt{x^5 + x^2 - 1} \Rightarrow \sqrt{1 + (f'(x))^2} = \sqrt{1 + x^5 + x^2 - 1} = \sqrt{x^5 + x^2}$

12.(b) $\frac{4}{9}\pi(27 - 2^{3/2})$

Here are the questions from the Dec 2004 MAT186H1F Final Exam, with answers:

1. [10 marks; 5 marks for each part.] Find the following:

   (a) $\dfrac{d}{dx}\left(e^{-3x}\sin(4x)\right)$

   (b) $\dfrac{dy}{dx}$ at the point $(x, y) = (1, -1)$ if

$$\sin^{-1}(y^2 - x) = x^2 y + 1$$

2. [10 marks: 5 marks for each part.] Find the following:

   (a) $F(x)$, if $F''(x) = 1 + x$ and $F(0) = -1, F'(0) = 3$.

   (b) $\displaystyle\int_1^{10} \frac{e^{-1/x}}{x^2}\, dx$

3. [8 marks; 4 marks for each part] The parts of this question are unrelated.

   (a) Find the maximum and minimum values of $f(x) = x^{1/3} + 2x^{4/3}$ on the closed interval $[-1, 1]$.

   (b) Four formulas are displayed below, in the column on the left. If the formula is incorrect, put the corrected formula in the corresponding space in the column on the right; if the formula is correct, leave the corresponding space on the right blank.

	Formula	Correction (if necessary)
(i)	$\displaystyle\int_b^a F'(x)\, dx = F(b) - F(a)$, if F' is a continuous function	$\displaystyle\int_b^a F'(x)\, dx = F(a) - F(b)$ or $\displaystyle\int_b^a F'(x)\, dx = F(b) - F(s)$
(ii)	$\displaystyle\frac{d}{dx}\left(\int_4^{x^2} f(z)\, dz\right) = 2xf(x)$, if f is a continuous function.	$\displaystyle\frac{d}{dx}\left(\int_4^{x^2} f(z)\, dz\right) = 2xf(x^2)$
(iii)	$\displaystyle\int \left(\frac{d}{dx}\sin\sqrt{x}\right) dx = \sin\sqrt{x}$	$\displaystyle\int \left(\frac{d}{dx}\sin\sqrt{x}\right) dx = \sin\sqrt{x} + C$
(iv)	$\displaystyle\int \frac{1}{x}\, dx = \ln x + C$	$\displaystyle\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$

4. [12 marks; 2 marks for each part] Let $f(x) = \sqrt{x^2 - 4x} + x$. Decide if the following statements about f are True or False:

(a) f is continuous for all values of x. True or False

(b) f has no critical points. True or False

(c) f has no inflection points. True or False

(d) There are no horizontal asymptotes to the graph of f. True or False

(e) There are no vertical asymptotes to the graph of f. True or False

(f) There are no vertical tangents to the graph of f. True or False

5. [10 marks] Find the following limits:

(a) [4 marks] $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$

(b) [6 HARD marks] $\lim_{x\to\infty} \left(\dfrac{\pi}{2} - \tan^{-1} x\right)^{1/\ln x}$

6. [10 marks] The velocity of a particle at time, t, is given by $v = t^2 - 5t + 6$, for $0 \le t \le 3$. Find:

(a) [4 marks] the average acceleration of the particle.

(b) [6 marks] the average speed of the particle.

7. [10 marks] The region R in the plane is bounded by the three curves with equations

$$y = \sqrt{x}, y = -x \text{ and } y = 2.$$

Find the volume of the solid obtained by rotating the region R around the x-axis.

8. [10 marks] The killer whale's tank at the Sea Life Aquarium is in the shape of a solid of revolution obtained by rotating the ellipse with equation

$$\frac{x^2}{30^2} + \frac{y^2}{15^2} = 1$$

around the y-axis, for $-15 \le y \le 0$, where distances are measured in metres.

Periodically the water in the tank must be completely pumped out and replaced with clean water. How much work is done in pumping all the water in the tank up to the top of the tank and out? (You may assume the acceleration due to gravity is $g = 9.8$ m/sec^2 and that the density of water is $\rho = 1000$ kg/m^3. Also, assume all whales have been removed from the tank before the pumping begins, and that the tank is full of water.)

9. [10 marks] Find the area of the region bounded by the curves with equations

$$y = x^3 \text{ and } y = 6x - x^2.$$

10. [10 marks] Find the length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$, for $1 \leq x \leq e$.

ANSWERS: 1(a) $-3e^{-3x}\sin(4x) + 4e^{-3x}\cos(4x)$ 1(b) 1/3

2(a) $\frac{1}{6}x^3 + \frac{1}{2}x^2 + 3x - 1$ 2(b) $e^{-1/10} - e^{-1}$

3(a) Min: $y = -\frac{3}{8}$; max: $y = 3$ 3(b) see filled-in chart above

4(a) False (b) False (c) True (d) False (e) True (f) False

5(a) 1/6 5(b) e^{-1} 6(a) -2 6(b) 29/18

7. $\frac{40}{3}\pi$ 8. $15^4\pi\rho g$ Joules 9. $\frac{253}{12}$ 10. $\frac{e^2-1}{2} + \frac{1}{4}$

FINAL EXAMINATION, APRIL 2004
First Year - CIV, CHE, IND, LME, MEC, MSE
MAT 186H1S - CALCULUS I

50 students wrote this exam. The marks ranged from 26% to 93%, and the class average was 62.7%

1. [10 marks; 5 marks for each part] Let the velocity v of a particle at time t be given by
$$v = t^2 - 6t + 8.$$
   (a) What is the average velocity of the particle between time $t = 0$ and time $t = 3$?
   (b) What is the total distance travelled by the particle between time $t = 0$ and time $t = 3$?

2. [20 marks; 5 marks for each part] Find the following limits:
   (a) $\lim\limits_{x\to 0} \dfrac{e^{-x} - 1 + x}{x^2}$
   (b) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} + \dfrac{1}{1-x})\right)$
   (c) $\lim\limits_{x\to 0} (1 - \sin x)^{3/x}$
   (d) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{1}{n}\left(\dfrac{i^2}{n^2} + 1\right)$

3. [10 marks; 5 marks for each part] Find the following:
   (a) [5 marks] $\displaystyle\int_0^8 \frac{x^2}{\sqrt{x+1}}\,dx.$
   (b) [5 marks] $F'(2)$, if $F(x) = \displaystyle\int_0^{\sqrt{x}} t^2 \sec^{-1} t\,dt.$

4. [10 marks] The radial probability density function for the ground state of the hydrogen atom is
$$P(r) = \left(\frac{4r^2}{a^2}\right) e^{-2r/a}, \text{ for } r \geq 0,$$
   where $a > 0$ is a constant. Sketch a graph of the function identifying its relative maximum and points of inflection.

5. [10 marks] A circular oil slick of uniform thickness is caused by a spill of 1m^3 of oil. The thickness of the oil slick is decreasing at the rate of 0.1 cm/hr. At what rate is the radius of the slick increasing when the radius is 8 m?

6. [10 marks] Find the area of the region bounded by the graphs of

$$y = 4 - x^2 \text{ and } y = x^2 + 4x - 2.$$

7. [10 marks] Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, y = x, x = 0, \text{ and } x = 1$$

around the line $y = -2$.

8. [10 marks] Find the area of the surface of revolution generated by revolving the curve

$$y = x^2 - \frac{1}{8}\ln x, \text{ for } 1 \leq x \leq 2$$

around the y-axis.

9. [10 marks] A hemispherical tank of radius 3 m is located with its flat side down atop a tower 15 m above the ground. How much work is required to fill this tank with oil of density ρ if the oil is to be pumped into the tank from ground level? (Use acceleration due to gravity $g = 9.8$ m/sec^2 and leave your answer in terms of ρ.)

ANSWERS: 1(a) 2 (b) 22/3 2(a) 1/2 (b) 1/2 (c) e^{-3} (d) 4/3
3(a) 992/15 (b) $\pi\sqrt{2}/8$ 4. max at $(a, 4/e^2)$; inflection points at $x = a(1 \pm 1/\sqrt{2})$
5. 0.256π m/hr 6. 64/3 7. $5\pi/6$ 8. $115\pi/6$ 9. $290.25\rho g\pi$ Joules

Here are the questions from the Final Exam in MAT186H1F (with answers) from December 2003.

1. [20 marks; 5 marks for each part.] Find the following limits:

   (a) $\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}$

   (b) $\lim_{x \to \frac{\pi}{2}^-} \left(\frac{2}{2x - \pi} + \sec x \right)$

   (c) $\lim_{x \to \infty} \left(1 - \frac{3}{x} \right)^{2x}$

   (d) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{e^{3i/n+2}}{n}$

2. [10 marks; 5 marks for each part] Let the velocity v of a particle at time t be given by

$$v = t^2 - 6t + 8.$$

   (a) What is the average velocity of the particle between time $t = 0$ and time $t = 4$?

   (b) What is the total distance travelled by the particle between time $t = 0$ and time $t = 4$?

3. [10 marks; 5 marks for each part] Find each of the following:

   (a) y as a function of x, if $\frac{dy}{dx} = \sin^2 x \cos x$ and $y = 2$ when $x = \frac{\pi}{2}$.

   (b) $\int_{-1}^{0} x^2 \sqrt{x+1}\, dx$. Hint: let $u = x + 1$.

4. [10 marks] The extension x (in cm) of a device designed to damp vibrations depends on time t (in seconds) as

$$x = e^{-t} \cos(8t), \text{ for } 0 \le t \le 1.$$

   Find all the critical points of x, and then determine the global extreme values of x. (Use your calculator to approximate your answers to 2 decimal places.)

5. [10 marks] A circular cylinder made of rubber is initially 150 mm long with a radius of 10 mm. The cylinder is then stretched so that its length increases at a rate of 3 mm/sec. Assuming that the volume of the cylinder remains constant, find the rate at which the radius is changing when the cylinder is 250 mm long.

6. [10 marks] Find the area of the region bounded by the two curves with equations

$$y = x^3 + 2x^2 + x - 2 \text{ and } y = x^3 + x^2 + 4x + 2.$$

7. [10 marks] Let $r > 0$; let V be the volume of the solid of revolution generated by revolving the region in the xy-plane bounded by

$$y = \frac{1}{1+x^2}, y = 0, x = r \text{ and } x = r+1$$

around the y-axis.

(a) [4 marks] Write down the integral that gives the value of V.

(b) [6 marks] Which value of r will maximize the value of V?

8. [10 marks] Find the area of the surface of revolution generated by revolving the curve with equation $y = \frac{x^2}{2}$, for $0 \leq x \leq 3$, around the y-axis.

9. [10 marks] Consider a spherical water tank with radius 3 m, and centre 15 m above the ground. How much work is required to fill this tank by pumping water up from ground level. (Assume the density of water is ρ, and leave your answer in terms of ρ.)

ANSWERS: 1.(a) $-1/3$ 1.(b) 0 1.(c) e^{-6} 1.(d) $(e^5 - e^2)/3$

2.(a) 4/3 2.(b) 8 3.(a) $y = (5 + \sin^3 x)/3$ 3.(b) 16/105

4. Critical points: $(0.38, -0.68)$ and $(0.77, 0.46)$; max: $x = 1$; min: $x = -0.68$

5. $-3\sqrt{15}/250$ mm/sec 6. 125/6 7.(a) $V = \int_r^{r+1} \frac{2\pi x}{1+x^2}\,dx$ 7.(b) $r = (-1+\sqrt{5})/2$

8. $2\pi(10\sqrt{10} - 1)/3$ 9. $540\pi\rho g$ (We will accept as correct the answer with or without g.)

University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, APRIL 2003
Year 1 – Programs 1, 2, 3, 4, 6, 7, 8, 9

MAT 186H1S
CALCULUS I
Examiner: D. Burbulla

Non-programmable calculator permitted; no other aids allowed.

TOTAL MARKS: 100.

1. [15 marks] Find the following derivatives:

   (a) [5 marks] $\dfrac{dy}{dx}$ if $y = e^{\sin x} - \cos e^x$.

   (b) [5 marks] $\dfrac{dy}{dx}$ if $y = \text{Tan}^{-1}x + \text{Tan}^{-1}\left(\dfrac{1}{x}\right)$.

   (c) [5 marks] $F'(2)$ if $F(x) = \displaystyle\int_0^{x^3} \sqrt{17+t^2}\,dt$.

2. [15 marks] Find the following limits:

   (a) [5 marks] $\displaystyle\lim_{x\to 0} \frac{1-2x^2-\cos(2x)}{x^2}$

   (b) [5 marks] $\displaystyle\lim_{x\to 1} \left(\frac{1}{\ln x} + \frac{1}{1-x}\right)$

   (c) [5 marks] $\displaystyle\lim_{x\to 0} \left(2-e^{-x}\right)^{\csc x}$

3. [15 marks] Find the following integrals:

   (a) [5 marks] $\displaystyle\int \frac{e^{3x}}{e^{3x}+10}\,dx$.

   (b) [5 marks] $\displaystyle\int_{-1}^{2} x^2(x^3+17)^{3/2}\,dx$.

   (c) [5 marks] $\displaystyle\int_1^{\infty} \left(\frac{1}{x^2}+\frac{4}{x^3}\right)\,dx$.

4. [15 marks] For this question, consider the curve with equation $x^{2/3} + y^{2/3} = 1$.

   (a) [5 marks] Show by differentiating implicitly that $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$.

   (b) [5 marks] Find the length of the curve from the point $(0, 1)$ to the point $(1, 0)$.

   (c) [5 marks] Find the volume of the solid obtained by revolving the curve for $-1 \leq x \leq 1$ around the x-axis.

5. [10 marks] A stone is dropped into a well and the sound of the stone striking the water is heard 3.1 sec later. If the speed of sound is 340 m/sec, how deep is the surface of the water in the well? (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

6. [10 marks] Sketch the graph of $f(x) = \text{Sec}^{-1}\sqrt{x^2+1}$, indicating all critical points, all inflection points, and all asymptotes, if any.

7. [10 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - A),$$

   where T is the temperature of an object at time t, A is the (constant) ambient temperature of the air surrounding the object, and k is a constant.

   A freshly baked pie is taken out of an oven with temperture 350° C and is placed on a table in a room with constant air temperature 20° C. If the pie cools by 100° C in 4 minutes, when will its temperature be 75° C?

8. [10 marks] A tank filled with water of density $\rho = 1000$ kg/m^3 has the shape of an inverted right circular cone, with radius 1 m at the top, and depth 3 m. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 2 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

ANSWERS: 1(a) $e^{\sin x}\cos x + e^x \sin e^x$ 1(b) 0 1(c) 108

2(a) 0 2(b) 1/2 2(c) e 3(a) $(1/3)\ln(e^{3x}+10)+c$ 3(b) 4202/15 3(c) 3

4(b) 3/2 4(c) $(32\pi)/105$ 5. 43.3 m

6. Graph is symmetric with respect to the $y-$axis; the right half of the graph, for $x \geq 0$, is the same as the graph of $\text{Tan}^{-1}x$.

7. 19.85 min 8. $2.75\rho g\pi$ Joules

Here are the questions from the Final Exam of MAT186H1F, December 2002, with answers. 463 students wrote this exam.

1. [30 marks; 5 marks for each part.] Find the following:

   (a) $\lim_{x\to 0} \dfrac{x - \mathrm{Sin}^{-1}x}{x^3}$

   (b) $\int_0^{\pi/2} \sin^3 x \cos x \, dx$

   (c) $\int_0^{\infty} \left(e^{-2x} + e^{-3x}\right) dx$

   (d) $\dfrac{dy}{dx}$ at the point $(x, y) = (-2, 2)$ if $y^2 = (4 + \sin(x + y))^{x^2-3}$.

   (e) $\lim_{x\to 0} \dfrac{\int_0^x \sin(t^2)\, dt}{x^3}$.

   (f) the average speed of a particle, with position $x = t^2 - 4t + 2$ at time t, on the interval $0 \le t \le 3$.

2. [12 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - A),$$

   where T is the temperature of an object at time t, A is the (constant) ambient temperature of the air surrounding the object, and k is a constant.

   A hot cup of coffee at 90° C is placed on a table in a room with constant air temperature 20° C. If the coffee cools to 50° C in 5 minutes, what will the temperature of the coffee be 2 minutes later? Sketch a graph of temperature, T, for $t \ge 0$.

3. [12 marks] Two poles are driven into the ground 5 m apart. One pole protrudes 3 m above the ground and the other pole 2 m above the ground. A single piece of rope is attached to the top of one pole, passed through a loop on the ground, pulled taut, and attached to the top of the other pole. Where should the loop be placed in order that the rope be as short as possible?

4. [12 marks; 6 marks for each part] Let $f(x) = \dfrac{1}{4}x^2$, for $0 \le x \le 2$. Find:

   (a) the volume of the solid of revolution obtained by rotating about the line $y = -1$ the region bounded by $y = f(x), y = 0, x = 0$ and $x = 2$.

   (b) the area of the surface of revolution obtained by rotating the curve $y = f(x)$ about the y-axis, for $0 \le x \le 2$..

5. [12 marks] A funnel in the shape of a right circular cone is 15 cm across the top and 30 cm deep. Water is flowing in at a rate of 80 mL/sec and flowing out the bottom at 15 mL/sec. At what rate is the surface of the liquid rising when the water fills the funnel to a depth of 20 cm? (NOTE: 1 mL = 1 cm^3.)

6. [12 marks] Sketch the graph of $f(x) = \text{Tan}^{-1}\sqrt{x^2 - 1}$, indicating all critical points, all inflection points, and all asymptotes, if any. (Be careful with domain and range!)

7. [10 marks] A tank filled with water of density $\rho = 1000$ kg/m^3 has the shape of a hemisphere, with radius 2 m. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 1 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

ANSWERS: 1(a) $-1/6$ 1(b) $1/4$ 1(c) $5/6$ 1(d) $\frac{1}{3} - \frac{16}{3}\ln 4$ 1(e) $1/3$ 1(f) $5/3$

2. 41.4° C 3. 3 m from the tall pole. 4(a) $\frac{26}{15}\pi$ 4(b) $\frac{8}{3}\pi(\sqrt{8} - 1)$ 5. $\frac{13}{5\pi}$ cm/sec

6. Domain: $|x| \geq 1$. Range: $0 \leq y < \pi/2$. Horizontal asymptote: $y = \pi/2$.

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}; \; f''(x) = \frac{1 - 2x^2}{x^2(x^2 - 1)^{3/2}}$$

Vertical tangents at $x = \pm 1$; no inflection points.

f decreasing for $x \leq -1$; increasing for $x \geq 1$; f concave down for all x in its domain.

7. $\frac{28}{3}\rho\pi g$ Joules, if you drew your hemispherical tank concave up; $12\rho\pi g$ Joules, if you drew your hemishperical tank concave down.

These are the questions from the Dec 2000 Final Exam in MAT186H1F. 374 students wrote it, with the marks ranging from 6% to 101% (there was one bonus mark available.) The average mark was 60.3%

1. [15 marks; avg: 12.1] Find the following:

   (a) [5 marks] $\lim_{x \to 0} \frac{x + \cos x - e^x}{x^2}$

   (b) [5 marks] $\lim_{x \to \infty} x^{1/x}$

   (c) [5 marks] $\frac{dy}{dx}$ at the point $(x, y) = (0, \pi)$ if $\sin y = xe^x$.

2. (avg: 9.6/15) (a) [10 marks] Given that f and h are continuous functions such that

$$\int_1^4 f(x)dx = -1, \int_4^9 f(x)dx = 5, \text{ and } \int_4^9 h(x)dx = 4,$$

   find the following:

   (i) $\int_1^9 -2f(x)dx$.

   (ii) $\int_4^9 (3f(x) - h(x))dx$.

   (iii) $\int_9^4 f(x)dx$.

   (iv) the average value of f on the interval $1 \leq x \leq 4$.

   (v) $\int_2^3 xf(x^2)dx$.

2.(b) [5 marks] Find $F'(2)$ if $F(x) = x \int_{2x}^{x^3-4} \frac{1}{1+\sqrt{t}}dt$.

3. [15 marks; avg: 11.8] Find the following:

   (a) [5 marks] $\int \frac{1}{x(\ln x)^2}dx$.

   (b) [5 marks] $\int x^3\sqrt{1+x^2}dx$.

   (c) [5 marks] $\int \left(e^x + e^{-x}\right)^2 dx$.

4. [15 marks; avg: 8.4] Let $f(x) = e^x - x^2$.

   (a) [4 marks] Use the tangent line approximation to $f(x)$ at $x = 0$ to approximate a solution to the equation $f(x) = 0$.

   (b) [6 marks] Starting with $x_1 = -1$, use Newton's method to approximate (correct to 3 decimal places) a solution to the equation $f(x) = 0$.

(c) [4 marks] Explain why there is only one negative solution x to the equation $f(x) = 0$. (BONUS mark: how many positive solutions are there?)

5. [10 marks; avg: 5.9] Let R be the region in the plane bounded by the curves $x = 0, y = \sqrt{x}$ and $y = 1$. (Draw a picture!) Let V be the volume of the solid generated by revolving R about the y-axis. Calculate V in *two* ways: by first using the method of cylindrical shells, and then by using the method of discs.

6. [10 marks; avg: 4.3] Train station A is at the intersection of two rail routes, one going east-west, and the other going north-south. When train E is just leaving station A and heading east at 50 km/hr, train N is exactly 40 km south of station A, heading north at 65 km/hr. Assuming the velocities of the two trains do not change, find the minimum possible distance between the trains.

7. [10 marks; avg: 2.8] A container in the form of an inverted right circular cone of radius 4 cm and height 10 cm is full of water. Water evaporates at a rate proportional to the area of the surface (of the water.) If the water level drops 1 cm in the first 5 days, how long will it take for the water to completely evaporate?

8. [10 marks; avg: 5.3] A tank filled with water of density $\rho = 1000$ kg/m^3 has the shape of a hemisphere (bottom half of a sphere) of radius 2 m. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 1 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

ANSWERS:

1(a) -1 1(b) 1 1(c) -1

2(a)(i) -8 2(a)(ii) 11 2(a)(iii) -5 2(a)(iv) $-1/3$ 2(a)(v) $5/2$ 2(b) $20/3$

3(a) $-\dfrac{1}{\ln(x)} + c$ 3(b) $\dfrac{1}{15}(3x^4 + x^2 - 2)\sqrt{1+x^2} + c$ 3(c) $-\dfrac{1}{2}e^{-2x} + \dfrac{1}{2}e^{2x} + 2x + c$

4(a) -1 4(b) $x_4 \approx -0.703467$ 4(c) no positive solutions

5. $V_{\text{shells}} = \int_0^1 2\pi x(1-\sqrt{x})\,dx = \dfrac{\pi}{5}$; $V_{\text{discs}} = \int_0^1 \pi(y^2)^2\,dy = \dfrac{\pi}{5}$

6. $\dfrac{400}{\sqrt{269}} \approx 24.3884$ km

7. 50 days 8. $\dfrac{28000}{3} g\pi$ Joules

University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, APRIL 2001
Year 1 – Programs 1, 2, 3, 4, 6, 7, 8, 9

MAT 186H1S
CALCULUS I
Examiner: D. Burbulla

INSTRUCTIONS

Non-programmable calculator permitted; no other aids allowed.

Present your solutions to all of the following questions in the exam booklets supplied. The marks for each question are indicated in parantheses.

TOTAL MARKS: 100.

1. [15 marks; each part is worth 5 marks] Find the following:
   (a) $\lim_{x\to 0} \frac{1-x^2-\cos x}{x^2}$
   (b) $\lim_{x\to 0^+} \left(\frac{1}{\sqrt{x}}+\ln x\right)$
   (c) $\lim_{x\to\infty} \left(e^{2x}+x\right)^{1/x}$
2. [15 marks; each part is worth 5 marks] Find the following:
   (a) the average value of $f(x)=x^3+x$ on the interval $1\le x\le 3$
   (b) the area of the region between the graphs of $f(x)=x^3$ and $g(x)=x$ for $0\le x\le 2$
   (c) $F'(-1)$ if $F(x)=\int_x^{x^2} \frac{1}{\sqrt{1+t^2}}\,dt$.

3. [15 marks; 5 marks for each part] Find the following:

   (a) $\int \frac{x^2+3x+7}{x+1}\,dx.$

   (b) $\int x^3\sqrt{1-x^2}\,dx.$

   (c) $\int \frac{1}{1+\sin x}\,dx.$

4. [15 marks] Let $f(x) = x^3 + x - 1$.

   (a) [6 marks] How many real solutions are there to the equation $f(x) = 0$? (Make sure you justify your answer.)

   (b) [9 marks] Approximate any one solution to the equation $f(x) = 0$ correct to 4 decimal places by using Newton's method.

5. [10 marks; 5 marks for each part] Consider the function $f(x) = x^2$ on the interval $0 \le x \le 1$. Find the following:

   (a) the volume of the solid obtained by revolving the function $f(x)$ around the line $y = -1$, for $0 \le x \le 1$

   (b) the area of the surface of revolution obtained by revolving the function $f(x)$ around the y-axis, for $0 \le x \le 1$

6. [10 marks] A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?

7. [10 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

   where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and $g = 9.8$ m/sec^2 is the acceleration due to gravity.

   A container in the form of an inverted right circular cone of radius 1 m and height 3 m is full of water. A circular plug of radius 1 cm is pulled open at the bottom of the container. How long will it take for the container to become completely empty of all water?

8. [10 marks] A tank filled with water of density $\rho = 1000$ kg/m^3 has the shape of a sphere of radius 3 m. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 1 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

Here are the questions from the Dec 2001 MAT186H1F exam. 409 students wrote this exam. The marks ranged from 15% to 98%, and the average exam mark was 60.8%

1. [16 marks] Find the following:

   (a) [5 marks] $\displaystyle\int_0^5 \frac{x}{\sqrt{x^2+16}}\,dx$

   (b) [5 marks] $\displaystyle\int_0^\infty \left(e^{-x}+e^{-2x}\right)\,dx$

   (c) [6 marks] $\displaystyle\lim_{x\to 0}\left(\frac{1}{2}+\frac{e^{3x}}{2}\right)^{\csc x}$

2.(a) [7 marks] At which point(s) on the circle with equation $x^2+y^2=1$ does the tangent line pass through the point $(3,0)$?

2.(b) [7 marks] Find y as a function of x if $\dfrac{dy}{dx} = y(2x+1)$ and $y=-3$ when $x=0$.

3. [14 marks] The position x of a particle at time t is given by $x = 15t^2 - 5t^3 - 6$, for $0 \le t \le 3$. Find the following:

   (a) [4 marks] the average velocity of the particle on the interval $0 \le t \le 3$.

   (b) [5 marks] the maximum speed of the particle on the interval $0 \le t \le 3$.

   (c) [5 marks] the average speed of the particle on the interval $0 \le t \le 3$.

4.(a) [6 marks] If Newton's method is used to approximate a solution to the equation $x^3+x-1=0$, and $x_1 = 0.5$, what is the value of x_3?

4.(b) [6 marks] Find the equations of all horizontal or vertical asymptotes (if any) to the graph of $y = \dfrac{x^2-1}{x^2+x-2}$.

5. [12 marks; 6 marks for each part] Let R be the region in the plane bounded by the curves $y=x^2$ and $y=2x-x^2$. (Draw a picture!) Find:

   (a) the volume of the solid obtained by revolving R about the x-axis.

   (b) the volume of the solid obtained by revolving R about the line $x=-1$.

6. [12 marks] A wire of length 100 cm is to be cut into two pieces. One piece will be bent into the shape of a circle; the other piece will be bent into the shape of a square. How should the wire be cut to

   (a) maximize the combined area of the two shapes?

   (b) mimimize the combined area of the two shapes?

7. [10 marks] The graph of the function $f(x)$ for $-3 \le x \le 5$ is:

Sketch the graph of $F(x) = \int_{-3}^{x} f(t)\,dt$ for $-3 \le x \le 5$, given that $F(0) = F(-3)$.

8. [10 marks] A tank filled with water of density $\rho = 1000$ kg/m^3 has the shape of an inverted right circular cone, with radius at the top 2 m and with vertical height 3 m. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 1 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

ANSWERS: 1(a) $\sqrt{41} - 4$ 1(b) 3/2 1(c) $e^{3/2}$

2(a) $(1/3, \pm 2\sqrt{2}/3)$ 2(b) $y = -3e^{x^2+x}$ 3(a) 0 3(b) 45 m/sec 3(c) 40/3 m/sec

4(a) $x_3 = 0.68318$ 4(b) horizontal asymptote $y = 1$; vertical asymptote $x = -2$

5(a) $\pi/3$ 5(b) π 6(a) whole wire bent into a circle 6(b) bend $100\pi/(4+\pi)$ cm into a circle; rest into a square 8. $7\rho\pi g$ Joules

University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, APRIL 2002
Year 1 – Programs 1, 2, 3, 4, 6, 7, 8, 9

MAT 186H1S
CALCULUS I
Examiner: D. Burbulla

Non-programmable calculator permitted. TOTAL MARKS: 100.

1. [15 marks; each part is worth 5 marks] Find the following:

   (a) $\displaystyle\lim_{x\to 0} \frac{1+2x-e^{2x}}{3x^2}$

   (b) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

   (c) $\displaystyle\lim_{x\to 0} \left(\frac{2}{3} + \frac{e^{-x}}{3}\right)^{\cot x}$

2. [15 marks; each part is worth 5 marks] The position x of a particle at time t is given by $x = 3t^2 - t^3 - 3$. Find the following:

   (a) the average velocity of the particle on the interval $-1 \le t \le 3$.

   (b) the maximum speed of the particle on the interval $-1 \le t \le 3$.

   (c) the average speed of the particle on the interval $-1 \le t \le 3$.

3. [15 marks; 5 marks for each part] Find the following:

   (a) $\displaystyle\int \frac{e^{3x}}{1+2e^{3x}}\,dx.$

   (b) $\displaystyle\int_0^4 \frac{x+7}{(x+1)^2}\,dx.$

   (c) $F'(\pi)$ if $\displaystyle F(x) = \int_0^{\sin x} \sqrt{1+t^3}\,dt.$

4. [10 marks; 5 marks for each part]

   (a) Find the area of the region bounded by the two parabolas with equations $y = x^2 - 1$ and $y = 1 - x^2$.

   (b) Find the equations of all horizontal or vertical asymptotes (if any) to the graph of $\displaystyle y = \frac{x^2-x-2}{x^2-1}$.

5. [10 marks; 5 marks for each part] Let $f(x) = x^{2/3}$. Find the following:

   (a) the volume of the solid obtained by revolving the region bounded by $y = f(x),\ y = 0,\ x = 0$ and $x = 1$ around the line $y = 3$.

   (b) the length of the curve $y = f(x)$ for $0 \le x \le 1$

6. [15 marks] Consider the parabola with equation $y = x^2$, and let $P = (a, a^2)$ for $a > 0$.

   (a) [5 marks] Find the equations of the tangent and normal lines to the parabola at the point P.

   (b) [10 marks] Let Q be the other point of intersection between the parabola and its normal line at P. (Draw a picture!) Find the value of a that minimizes the y-coordinate of Q.

7. [10 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

   where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and $g = 9.8$ m/sec^2 is the acceleration due to gravity.

   A container in the form of an inverted right circular cone of radius 2 m (at the top) and height 3 m is full of water. A circular plug of radius 1 cm = 0.01 m is pulled open at the bottom of the container. How long will it take for the container to become completely empty of all water?

8. [10 marks] A hemispherical tank (the bottom half of a sphere) of radius 3 m is filled with water of density $\rho = 1000$ kg/m^3. Find the work done in pumping all of the water out of the tank and up to a horizontal pipe 1 m above the top of the tank. (Use acceleration due to gravity $g = 9.8$ m/sec^2.)

ANSWERS: 1(a) $-2/3$ 1(b) 0 1(c) $e^{-1/3}$ 2(a) -1 2(b) 9 2(c) 3

3(a) $(1/6)\ln(1 + 2e^{3x}) + c$ 3(b) $24/5 + \ln 5$ 3(c) -1

4(a) $8/3$ 4(b) HA: $y = 1$; VA: $x = 1$ 5(a) 9.96 approx 5(b) 1.436 approx

6(a) tangent: $y = 2ax - a^2$ normal: $y = -1/(2a)x + 1/2 + a^2$ 6(b) $a = 1/\sqrt{2}$

7. 1.74 hr, approx. 8. $38.25\pi\rho g$ Joules