UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2011 EXAMINATIONS

MAT335H1F Solutions
Chaos, Fractals and Dynamics
Examiner: D. Burbulla

Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

General Comments:

1. Many students used bad logic, especially in proffered solutions to Questions 2 and 3.

2. A simpler solution (than mine) to Question 3, part (c) was supplied: note that

$$F : [1/4, 3] \longrightarrow [-2, 1/4] \text{ and } F : [-2, 1/4] \longrightarrow [1/4, 3].$$

   Then for any odd value q,

$$F^q(x) = x \Rightarrow x = \frac{1}{4},$$

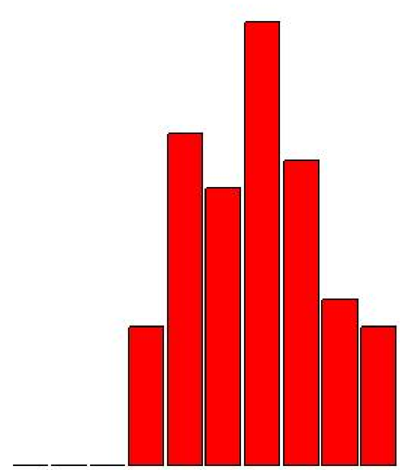
   which is the fixed point of F. So the cycle has prime period $q = 1$.

3. In Question 5, if you want to describe the action of A_1 and A_2 geometrically, you have to find the fixed point of each transformation; only one of them has fixed point $(0, 0)$.

4. Questions 2 and 5 were the least popular questions.

Breakdown of Results: 65 students wrote this exam. The marks ranged from 35% to 98%, and the average was 63.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

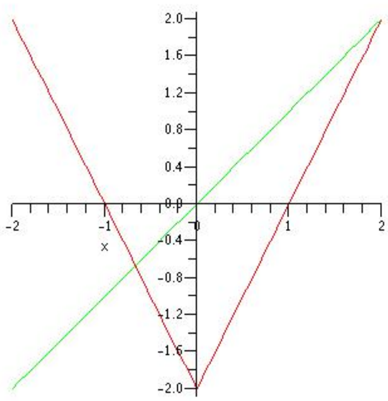
Grade	%	Decade	%
		90-100%	7.7%
A	16.9%	80-89%	9.2%
B	16.9%	70-79%	16.9%
C	24.6%	60-69%	24.6%
D	15.4%	50-59%	15.4%
F	26.2%	40-49%	18.5%
		30-39%	7.7%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



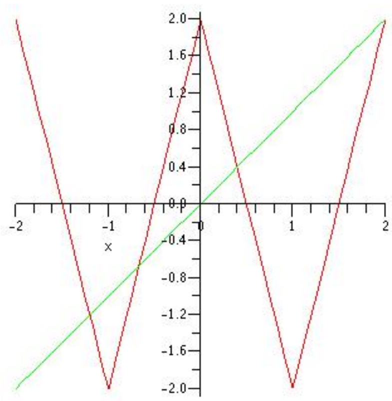
1. [20 marks] Let $V : [-2, 2] \longrightarrow [-2, 2]$ by $V(x) = 2|x| - 2$.

   (a) [5 marks] Plot the graphs of V and V^2.

   Solution: with the line $y = x$ also drawn in.



graph of V



graph of V^2

   (b) [5 marks] Find all the fixed points and 2-cycles of V and determine if they are attracting or repelling.

   Solution: For fixed points, $V(x) = x$:

$$x \leq 0 \Rightarrow -2x - 2 = x \Rightarrow x = -2/3 \text{ and } x > 0 \Rightarrow 2x - 2 = x \Leftrightarrow x = 2$$

   Both fixed points are repelling since $|V'(-2/3)| = |V'(2)| = 2 > 1$.

   For 2-cycles, $V^2(x) = x$ but $V(x) \neq x$:

$$-2 < x < -1 \Rightarrow -4x - 6 = x \Rightarrow x = -6/5$$

   and

$$0 < x < 1 \Rightarrow -4x + 2 = x \Rightarrow x = 2/5$$

   So the only 2-cycle is $-6/5$ and $2/5$, which is repelling since

$$|V'(-6/5)|\,|V'(2/5)| = 2^2 = 4 > 1.$$

(c) [10 marks] Let $T : [0,1] \longrightarrow [0,1]$ by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} < x \leq 1 \end{cases} ;$$

let $h : [0,1] \longrightarrow [-2,2]$ by $h(x) = -4x+2$. Prove that h is a conjugacy between T and V.

Solution: You have to check that h is 1-1, onto, continuous, with continuous inverse—which are all obvious since h is linear—and that

$$h \circ T = V \circ h.$$

This last equation should be verified:

$$\begin{aligned} h(T(x)) &= -4T(x)+2 \\ &= \begin{cases} -4(2x)+2, & \text{if } 0 \leq x \leq 1/2 \\ -4(2-2x)+2, & \text{if } 1/2 < x \leq 1 \end{cases} \\ &= \begin{cases} -8x+2, & \text{if } 0 \leq x \leq 1/2 \\ 8x-6, & \text{if } 1/2 < x \leq 1 \end{cases} \end{aligned}$$

$$\begin{aligned} V(h(x)) = V(-4x+2) = 2|-4x+2|-2 &= \begin{cases} 2(-4x+2)-2, & \text{if } -4x+2 \geq 0 \\ 2(4x-2)-2, & \text{if } -4x+2 < 0 \end{cases} \\ &= \begin{cases} -8x+2, & \text{if } x \leq 1/2 \\ 8x-6, & \text{if } x > 1/2 \end{cases} \end{aligned}$$

Thus $V \circ h = h \circ T$.

2. [20 marks] This question has five parts.

(a) [3 marks] Define: the subset D is dense in X.

Solution: the definition on page 114 of Devaney is

> D is dense in X if for any point $x \in X$ there is a point $d \in D$ arbitrarily close to x.

I would also accept any of these two *equivalent* conditions:

> D is dense in X if for any point $x \in X$ there is a sequence $\{d_n\}$, consisting of points in D, that converges to x.
>
> D is dense in X if for every open subset A of X, $D \cap A \neq \phi$.

(b) [7 marks] Prove that the periodic points of σ are dense in Σ.

Solution: Let $\mathbf{s} = (s_0 s_1 \ldots s_n s_{n+1} \ldots)$ be an arbitrary sequence in Σ, let $\epsilon > 0$. Pick n such that $1/2^n < \epsilon$; let $\mathbf{t} = (\overline{s_0 s_1 \ldots s_n})$. Then $\mathbf{t}$ is a periodic point of σ and, by the Proximity Theorem,

$$d[\mathbf{s}, \mathbf{t}] \leq 1/2^n < \epsilon.$$

(c) [6 marks] Define the Cantor middle-thirds set, K. What is its fractal dimension?

Solution: here's a recursive definition for K.

1. Start with the interval $[0, 1]$.
2. Remove the middle third $(1/3, 2/3)$, leaving two closed intervals left, $[0, 1/3]$ and $[2/3, 1]$, each of length $1/3$.
3. Repeat this process: remove the open middle third from each of the previous closed intervals.
4. K is the set of points remaining in $[0, 1]$ in the limit as this process is repeated over and over without end.

The fractal dimension of K is

$$\frac{\log 2}{\log 3} = 0.630929753\ldots$$

(d) [2 marks] Show that K is not dense in $[0, 1]$.

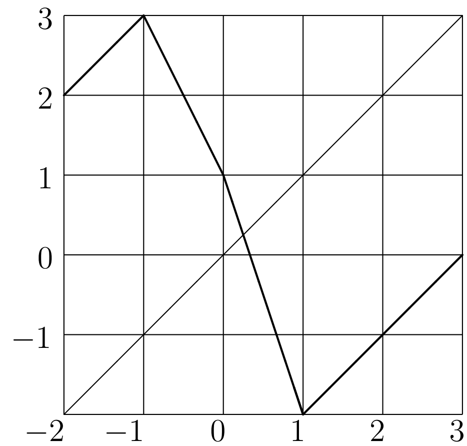
Solution: by definition of the Cantor middle-thirds set, $(1/3, 2/3) \cap K = \phi$; so there is an open interval that does not intersect K.

(e) [2 marks] Show that the complement of K is dense in $[0, 1]$.

Solution: since the Cantor middle-thirds set is totally disconnected, it contains no open interval. Thus every open interval $(a, b) \subset [0, 1]$ must intersect the complement of K. That is

$$(a, b) \not\subset K \Rightarrow (a, b) \cap ([0, 1] - K) \neq \phi.$$

3. [20 marks] The graph of $F : [-2,3] \longrightarrow [-2,3]$ is shown below, along with the line $y = x$.



(a) [5 marks] Show that -2 is on a 6-cycle for F.

Solution: the 6-cycle for F is

$$-2 \to 2 \to -1 \to 3 \to 0 \to 1 \to -2$$

(b) [5 marks] Explain why F has cycles with prime period p for any even number p.

Solution: by Sarkovskii's Theorem F will have cycles of prime period p for any number p after 6 in the Sarkovskii ordering. But these numbers comprise all the even numbers.

(c) [10 marks] Prove that F has no cycles of any odd prime period $q > 1$.

Solution: Observe that $F : [-2,0] \longrightarrow [1,3]$ and $F : [1,3] \longrightarrow [-2,0]$. So for any odd value $q > 1$

$$F^q([-2,0]) = [1,3] \text{ and } F^q([1,3]) = [-2,0].$$

If F^q has a fixed point it must be in the interval $(0,1)$. Suppose the fixed point is x; then the q-cycle for F must be

$$x, H(x), H^2(x), \ldots, H^{q-1}(x),$$

for $H = F$, restricted to the interval $(0,1)$, with each point

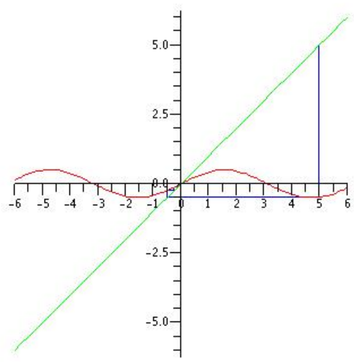
$$x, H(x), H^2(x), \ldots, H^{q-1}(x) \in (0,1).$$

Since H is 1-1, the only solution to $H^q(x) = x$ is the fixed point of H in $(0,1)$, and so the cycle has prime period 1.

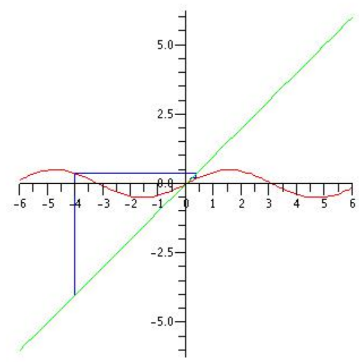
4. [20 marks] For $c \neq 0$, let $F_c : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_c(x) = c \sin x$.

(a) [4 marks] Show that for $|c| < 1$, $x = 0$ is the only fixed point of F_c and its basin of attraction is $\mathbb{R} = (-\infty, \infty)$.

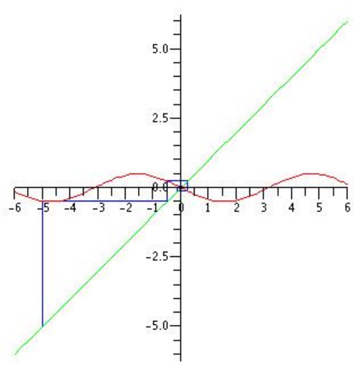
Solution: for $0 < |c| < 1$ the amplitude of $F_c(x) = c \sin x$ is $|c| < 1$, so the graph of $F_c(x)$ will never intersect the line $y = x$ for $x \neq 0$. Then use graphical analysis to show that for $x_0 \in \mathbb{R}$, $x_n \to 0$.



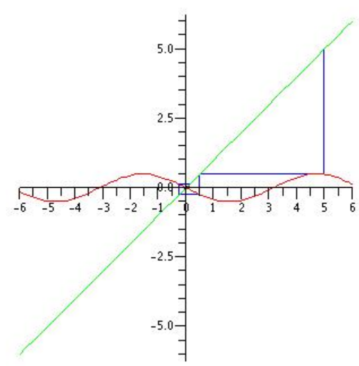
orbit of 5 under $c \sin x, 0 < c < 1$



orbit of -4 under $c \sin x, 0 < c < 1$



orbit of -5 under $c \sin x, -1 < c < 0$



orbit of 5 under $c \sin x, -1 < c < 0$

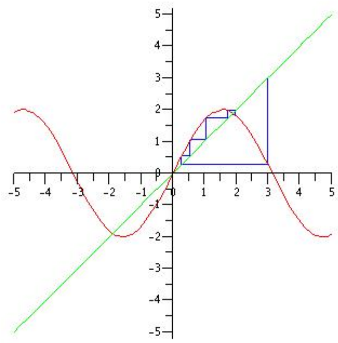
(b) [4 marks] Calculate the Schwarzian derivative of F_c and show it is negative.

Solution: for $\cos x \neq 0$, $S(F_c)(x) =$

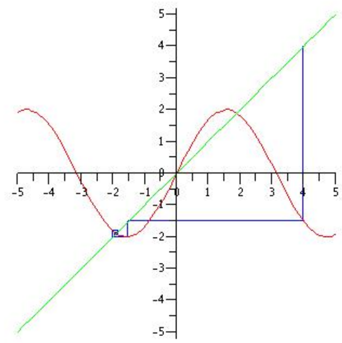
$$\frac{F_c'''(x)}{F_c'(x)} - \frac{3}{2}\left(\frac{F_c''(x)}{F_c'(x)}\right)^2 = \frac{-c\cos x}{c\cos x} - \frac{3}{2}\left(\frac{-c\sin x}{c\cos x}\right)^2 = -1 - \frac{3}{2}\tan^2 x < 0.$$

(c) [4 marks] Give a graphical example of a fixed point of F_c for which the immediate basin of attraction does not extend to infinity.

Solution: pick $c > 1$ such that $y = F_c(x)$ intersects the line $y = x$ with slope between -1 and 0. For the following graphs, $c = 2$.



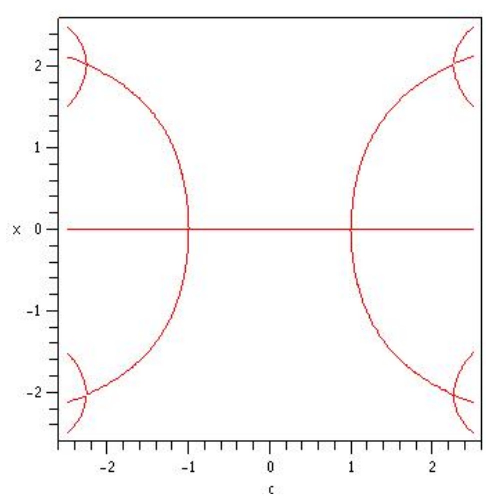
orbit of 3 under $2 \sin x$



orbit of 4 under $2 \sin x$

The immediate basin of attraction of the fixed point of $2 \sin x$ close to 2, is $(0, \pi)$. Note: if you take $c = \pi/2$, then the fixed points of $F_{\pi/2}$ are exactly $p = \pm\pi/2$, which are also critical points of F_c. The immediate basin of attraction of the fixed point $p = \pi/2$ is also $(0, \pi)$:

(d) [8 marks] Below is the bifurcation diagram for F_c, for $|c| < 2.5$, $|x| < 2.5$.



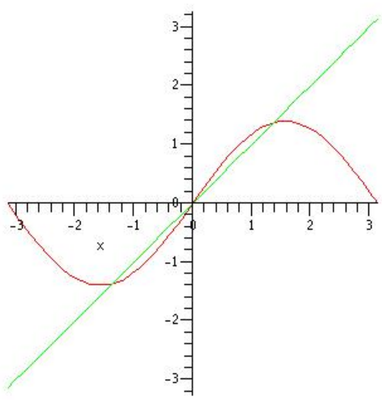
Classify each node in this diagram as a tangent (or saddle-node) bifurcation, a period-doubling bifurcation, or neither.

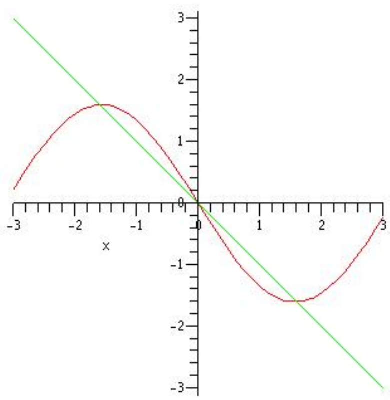
Solution: the fixed point $x = 0$ is attracting $\Leftrightarrow -1 < c < 1$. The nodes on the c-axis are $(\pm 1, 0)$. The four other nodes are $(\pm a, \pm b)$ such that $a > 0, b > 0$ and

$$\left\{\begin{array}{rcl} F_a(b) & = & b \\ F_a'(b) & = & -1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} a\sin b & = & b \\ a\cos b & = & -1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \tan b & = & -b \\ a^2 & = & b^2+1 \end{array}\right. ;$$

whence $a \simeq 2.26, b \simeq 2.03$. But you don't really need these values.

node $(1,0)$**:** neither, since $x=0$ is attracting for $c<1$, repelling for $c>1$; and for $c>1$ two new attracting fixed points appear. See the graph below, on the left:





node $(-1,0)$**:** period-doubling, since $x=0$ is attracting for $c>-1$, repelling for $c<-1$; and for $c<-1$ an attracting 2-cycle appears, namely the two solutions to $c\sin x=-x$. See the graph above, on the right.

nodes $(a,\pm b)$**:** period-doubling, since $F_a'(\pm b)=-1$

nodes $(-a,\pm b)$**:** neither, since the attracting 2-cycle for $-a<c<-1$ becomes repelling; but for $c<-a$ two new attracting 2-cycles show up. To see this, calculate orbits of critical points $x=\pm\pi/2$ under F_c, using $c=-1.5$ and $c=-2.4$:

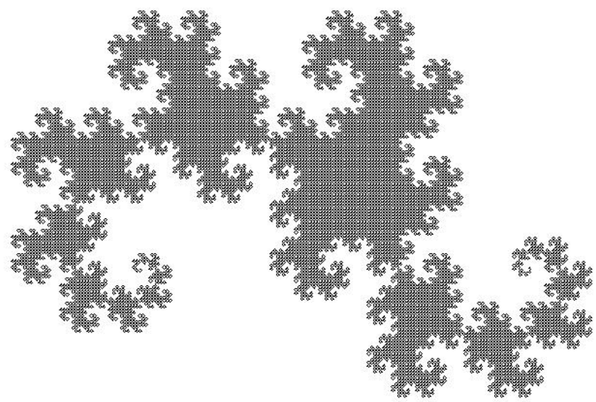
c	x_0	orbit is attracted to the 2-cycle
-1.5	$\pm\pi/2$	$1.495781568\ldots, -1.495781568\ldots$
-2.4	$\pi/2$	$-2.396065934\ldots, 1.628061364\ldots$
-2.4	$-\pi/2$	$2.396065934\ldots, -1.628061364\ldots$

5. [20 marks] The following iterated function system

$$A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

generates the following fractal, known as the dragon curve:



(a) [10 marks] Here is one way to generate the dragon curve:

Step 0: Draw the line segment I which joins the points $(0,0)$ and $(1,0)$.

Step 1: Replace I by the two line segments $A_1(I)$ and $A_2(I)$.

Step 2: Replace the two line segments of Step 1 by the four line segments

$$A_1 \circ A_1(I), A_1 \circ A_2(I), A_2 \circ A_1(I) \text{ and } A_2 \circ A_2(I).$$

Step k: Replace each line segment of Step $k-1$ by its images under A_1 and A_2.

Draw Steps 0 through 3 of this process.

Solution: direct computational approach. In simplified form

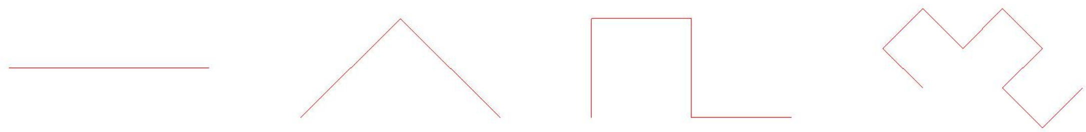
$$A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x - y \\ x + y \end{pmatrix}, \quad A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 - x - y \\ x - y \end{pmatrix}.$$

To calculate A_1 or A_2 of a line segment you only need to calculate A_1 and A_2 of the end points. For Step 1:

$$A_1(I): \quad A_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$A_2(I): \quad A_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; A_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Here are the four graphs:



Step 0 Step 1 Step 2 Step 3

Here are the rest of the calulations. For Step 2:

$$A_1 \circ A_1(I): \quad A_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ A_1\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$A_2 \circ A_1(I): \quad A_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ A_2\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$A_1 \circ A_2(I): \quad A_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},\ A_1\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$A_2 \circ A_2(I): \quad A_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},\ A_2\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

For Step 3:

$$A_1 \circ A_1 \circ A_1(I): \quad A_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ A_1\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}$$

$$A_2 \circ A_1 \circ A_1(I): \quad A_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ A_2\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/4 \end{pmatrix}$$

$$A_1 \circ A_2 \circ A_1(I): \quad A_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},\ A_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$$

$$A_2 \circ A_2 \circ A_1(I): \quad A_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},\ A_2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$$

$$A_1 \circ A_1 \circ A_2(I): \quad A_1\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},\ A_1\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}$$

$$A_2 \circ A_1 \circ A_2(I): \quad A_2\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},\ A_2\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/4 \end{pmatrix}$$

$$A_1 \circ A_2 \circ A_2(I): \quad A_1\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},\ A_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$$

$$A_2 \circ A_2 \circ A_2(I): \quad A_2\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},\ A_2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$$

(b) [5 marks] Calculate the fractal dimension of the dragon curve.

Solution: at each step you double the number of line segments and the magnification factor is $\sqrt{2}$, so the fractal dimension of the dragon curve is

$$\frac{\log 2}{\log \sqrt{2}} = \frac{\log 2}{\frac{1}{2}\log 2} = 2.$$

It is actually a space filling curve.

(c) [5 marks] Describe another algorithm that generates the dragon curve.

Solution: I'll accept almost any alternative description as long as it is mathematically concrete, not exactly the same as part (a), and describes clearly a recursive procedure. Here are some possibilities:

Algorithm 1: Play the chaos game. That is, start with a point p_0 in the plane. Pick A_1 or A_2 and apply it to p_0 to obtain p_1. Now pick A_1 or A_2 and apply it to p_1 to obtain p_2. Continue recursively in this way: to obtain p_{k+1} randomly pick either A_1 or A_2 and apply it to p_k. The orbit of p_0, namely $p_0, p_1, p_2, \ldots, p_k, \ldots$ as $k \to \infty$, is attracted to the dragon curve.

Algorithm 2: Interpret each function A_1, A_2 geometrically.

1. A_1 is a rotation or $45°$ around its fixed point $(0, 0)$ followed by a contraction of $\beta = 1/\sqrt{2}$ towards the fixed point $(0, 0)$.
2. A_2 is rotation of $135°$ around its fixed point $(3/5, 1/5)$ followed by a contraction of $\beta = 1/\sqrt{2}$ towards the fixed point $(3/5, 1/5)$.

Then proceed as in part (a), performing the above two operations on each line segment of the previous stage.

Algorithm 3: Alternating triangles. Start with the line segment I and on it construct the two sides of an isosceles right triangle with I as its hypotenuse. At each subsequent step construct an isosceles right triangle on each segment of the previous stage, alternating the side on which the triangle appears, as you go: that is, starting at $(0, 0)$, first triangle is on the left, next triangle is on the right, and so on.

6. [20 marks] Let $Q_c : \mathbb{C} \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$.

(a) [4 marks] Define the Mandelbrot set, $\mathcal{M}$.

Solution: the actual definition on page 249 of Devaney is

> $\mathcal{M}$ consists of all c-values for which the filled Julia set K_c is connected.

I would also accept this equivalent statement:

$$\mathcal{M} = \{c \in \mathbb{Z} \mid \text{the orbit of } 0 \text{ under } Q_c \text{ is bounded}\}$$

(b) [4 marks] Show that the orbit of 0 under Q_{-2} is eventually fixed. Is this fixed point attracting or repelling? Is $-2 \in \mathcal{M}$?

Solution: $Q_{-2}(0) = -2$; $Q_{-2}(-2) = 2$; $Q_{-2}(2) = 2$; so 0 is eventually fixed. $z = 2$ is a repelling fixed point since

$$|Q'_{-2}(2)| = 4 > 1.$$

And $-2 \in \mathcal{M}$, since the orbit of 0 under Q_{-2} is bounded.

(c) [6 marks] Show that the orbit of 0 under Q_i is eventually periodic. Is this cycle attracting or repelling? Is $i \in \mathcal{M}$?

Solution:

$$Q_i(0) = i;\ Q_i(i) = -1 + i;\ Q_i(-1 + i) = -i;\ Q_i(-i) = -1 + i;$$

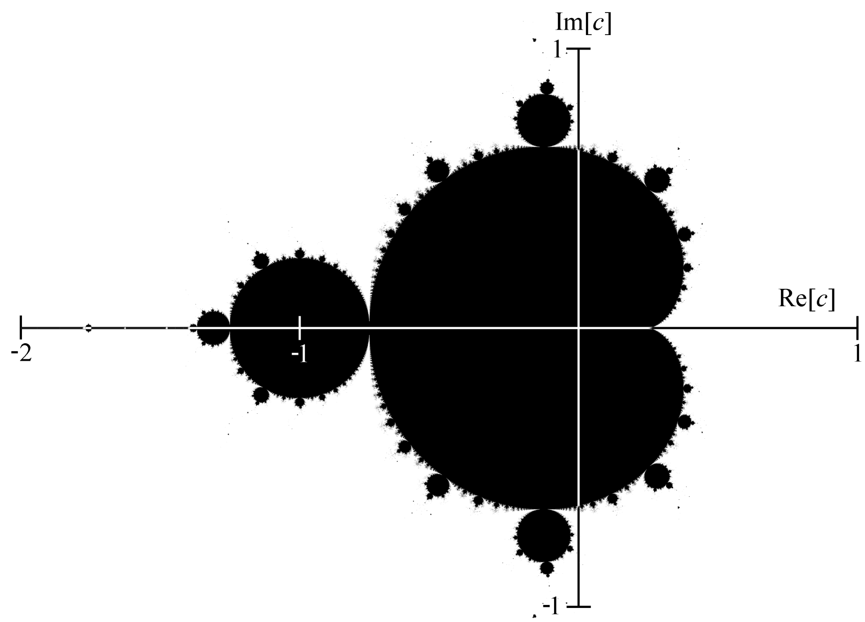
so 0 is eventually attracted to a 2-cycle. This 2-cycle is repelling since

$$|Q'_{-2}(-1 + i)||Q'_{-2}(-i)| = 2\sqrt{2} \cdot 2 > 1.$$

So $i \in \mathcal{M}$, since the orbit of 0 under Q_i is bounded.

(d) [2 marks] With respect to the following image of the Mandelbrot set, locate both -2 and i.

Solution:



Note: neither -2 nor i can be in a bulb of the Mandelbrot set, for then their orbits would eventually end up on an attracting cycle; nor on the boundary of a bulb of the Mandelbrot set, for then they would both be neutral periodic points. So by elimination the two points -2 and i must be on the antennae of the Mandelbrot set.

(e) [4 marks; 2 marks each] Let K_c be the filled Julia set of Q_c; let J_c be the Julia set of Q_c. Indicate whether the following statements are True or False, and give a brief justification for your choice.

I. K_{-2} is totally disconnected.

Solution: False. See the definition of $\mathcal{M}$ above. Since $-2 \in \mathcal{M}$, K_{-2} is connected.

II. $K_{-2} = J_{-2}$

Solution: True. We did this one in class. K_{-2} is the line segment along the real axis joining -2 to 2. That is, K_{-2} has no interior (in the complex plane) so its boundary, J_{-2}, is equal to itself.