## MAT188H1S - Linear Algebra

## Solutions to Term Test - Friday, March 13, 2020

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

## **Comments:**

- All in all, the results on this test were quite good.
- The only question with a failing average was Question 7, which involved proofs. No surprise!
- The computational question that caused the most difficulty was Question 6, finding the eigenvalues and eigenvectors of a  $3 \times 3$  matrix.

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**Breakdown of Results:** 33 students wrote this test. The marks ranged from 40% to 96.25%, and the average was 69.4%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | %     | Decade   | %     |
|-------|-------|----------|-------|
|       |       | 90-100%  | 6.1%  |
| А     | 18.2% | 80 - 89% | 12.1% |
| В     | 33.3% | 70-79%   | 33.3% |
| C     | 33.3% | 60-69%   | 33.3% |
| D     | 3.0%  | 50-59%   | 3.0%  |
| F     | 12.1% | 40-49%   | 12.1% |
|       |       | 30-39%   | 0.0%  |
|       |       | 20-29%   | 0.0%  |
|       |       | 10 -19%  | 0.0%  |
|       |       | 0-9%     | 0.0%  |



1. [2 marks for each part; avg: 8.9/10] Let  $\vec{u} = \begin{bmatrix} 1\\ 3\\ -1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 6\\ -1\\ 4 \end{bmatrix}$ . Find the following:

(a)  $\vec{u}\cdot\vec{v}$ 

Solution:

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 1\\3\\-1 \end{bmatrix} \cdot \begin{bmatrix} 6\\-1\\4 \end{bmatrix} = 6 - 3 - 4 = -1$$

(b)  $\|\vec{v}\|$ 

Solution:

$$\|\vec{v}\| = \left\| \begin{bmatrix} 6\\ -1\\ 4 \end{bmatrix} \right\| = \sqrt{36 + 1 + 16} = \sqrt{53}$$

(c) the projection of  $\vec{u}$  on  $\vec{v}$ .

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = -\frac{1}{53} \begin{bmatrix} 6\\ -1\\ 4 \end{bmatrix}.$$

(d)  $\vec{u} \times \vec{v}$ 

Solution:

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1\\3\\-1 \end{bmatrix} \times \begin{bmatrix} 6\\-1\\4 \end{bmatrix} = \begin{bmatrix} 11\\-10\\-19 \end{bmatrix}$$

(e) parametric equations of the line with vector equation  $\vec{x} = \vec{u} + t \vec{v}$ , where t is a parameter. Solution:

- 2. [avg: 8.5/10] Consider the three points P, Q, R with coordinates P(5, 1, -1), Q(2, 1, 0) and R(7, 3, -3).
  - (a) [7 marks] Find the scalar equation of the plane that passes through the three points P, Q and R.

**Solution:** take the normal of the plane to be  $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ . We have

$$\overrightarrow{PQ} = \begin{bmatrix} -3\\0\\1 \end{bmatrix}, \ \overrightarrow{PR} = \begin{bmatrix} 2\\2\\-2 \end{bmatrix}, \ \overrightarrow{n} = \begin{bmatrix} -3\\0\\1 \end{bmatrix} \times \begin{bmatrix} 2\\2\\-2 \end{bmatrix} = \begin{bmatrix} -2\\-4\\-6 \end{bmatrix}.$$

Pick Q as a point on the plane; its scalar equation is

$$-2x_1 - 4x_2 - 6x_3 = -4 - 4 + 0 = -8 \Leftrightarrow x_1 + 2x_2 + 3x_3 = 4.$$

(b) [3 marks] What is the area of the triangle with vertices P, Q, R?

**Solution:** the area of a triangle is half the area of the parallelogram formed by two sides of the triangle. Thus

area of 
$$\Delta PQR = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2} \left\| \begin{bmatrix} -2\\ -4\\ -6 \end{bmatrix} \right\| = \frac{2}{2} \left\| \begin{bmatrix} 1\\ 2\\ 3 \end{bmatrix} \right\| = \sqrt{14}.$$

3. [avg: 6.8/10] Consider the two lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  in space defined by

$$\mathcal{L}_1: \left\{ \begin{array}{rrrr} x &=& 1 &+& t \\ y &=& -1 &-& 2t \\ z &=& 1 &+& t \end{array} \right. \text{ and } \mathcal{L}_2: \left\{ \begin{array}{rrrr} x &=& 6 &-& 2s \\ y &=& 5 &+& 4s \\ z &=& 4 &-& 2s \end{array} \right.,$$

where s and t are parameters.

(a) [5 marks] Explain why  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are parallel, distinct lines.

## Solution:

• Take 
$$\vec{d_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$
 as the direction vector of  $\mathcal{L}_1$ .  
• Take  $\vec{d_2} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$  as the direction vector of  $\mathcal{L}_2$ .

- Since  $\vec{d_2} = -2\vec{d_1}$ , the direction vectors are parallel, and so the lines are parallel too.
- But the lines are distinct since the point (6, 5, 4) is on  $\mathcal{L}_2$  but can't be on  $\mathcal{L}_1$ , because every point on  $\mathcal{L}_1$  has x = z.

(b) [5 marks] Find the constant (perpendicular) distance between the two lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$ .

**Solution:** since the lines are parallel this is equivalent to finding the minimum distance from any point on  $\mathcal{L}_2$  to  $\mathcal{L}_1$ . Take the point  $P_2(6, 5, 4)$  on  $\mathcal{L}_2$ ; take the point  $P_1(1, -1, 1)$  on  $\mathcal{L}_1$ . Then the minimum distance from  $P_2$  to  $\mathcal{L}_1$  is

$$D = \|\overrightarrow{P_1P_2} - \operatorname{proj}_{\vec{d_1}}(\overrightarrow{P_1P_2})\| = \left\| \overrightarrow{P_1P_2} - \frac{\overrightarrow{P_1P_2} \cdot \vec{d_1}}{\|\vec{d_1}\|^2} \vec{d_1} \right\|$$
$$= \left\| \begin{bmatrix} 5\\6\\3 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1\\-2\\1 \end{bmatrix} \right\|$$
$$= \frac{1}{3} \left\| \begin{bmatrix} 17\\14\\11 \end{bmatrix} \right\|$$
$$= \frac{\sqrt{606}}{3} = \sqrt{\frac{202}{3}} \approx 8.2$$

4. [avg: 7.0/10] Let  $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  be the linear transformation defined by

$$T\left(\left[\begin{array}{c} x_1\\ x_2 \end{array}\right]\right) = \left[\begin{array}{c} 2x_1 + 4x_2\\ -3x_1 + 2x_2 \end{array}\right].$$

(a) [5 marks] Draw the image of the unit square<sup>1</sup> under T and label all four vertices.

Solution: 
$$T(\vec{0}) = \vec{0}$$
 and  
 $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-3\end{bmatrix}; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\2\end{bmatrix};$   
 $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\-1\end{bmatrix}.$   
(0,0)  
(0,-1)  
(2,-3)

(b) [5 marks] Find  $T^{-1}\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$ .

**Solution:** let the matrix of T be

$$A = \left[ \begin{array}{cc} 2 & 4 \\ -3 & 2 \end{array} \right]$$

Then the matrix of  $T^{-1}$  is

$$A^{-1} = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{16} \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix},$$

by the formula for the inverse of a  $2 \times 2$  matrix, as proved in class. Then

$$T^{-1}\left(\left[\begin{array}{c}x_1\\x_2\end{array}\right]\right) = \frac{1}{16}\left[\begin{array}{c}2&-4\\3&2\end{array}\right]\left[\begin{array}{c}x_1\\x_2\end{array}\right] = \frac{1}{16}\left[\begin{array}{c}2x_1-4x_2\\3x_1+2x_2\end{array}\right]$$

<sup>&</sup>lt;sup>1</sup>The unit square is the square with the four vertices (0,0), (1,0), (0,1), (1,1).

5. [avg: 7.8/10] Solve the following system of linear differential equations

$$\begin{array}{rcrcrcrcrc} f_1' &=& -f_1 &+& 5f_2\\ f_2' &=& f_1 &+& 3f_2 \end{array}$$

for  $f_1$  and  $f_2$  as functions of x if  $f_1(0) = 1$  and  $f_2(0) = -1$ .

**Solution:** the coefficient matrix is  $A = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$ , for which

$$\det(\lambda I - A) = (\lambda + 1)(\lambda - 3) - 5 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2).$$

The eigenvalues of A are  $\lambda_1 = 4$  and  $\lambda_2 = -2$ , the roots of the characteristic polynomial. To find the eigenvectors, find a (simple) non-zero solution  $\vec{v}$  to the homogeneous system  $(\lambda I - A)\vec{v} = \vec{0}$ :

For eigenvalue  $\lambda_1 = 4$ :  $\begin{bmatrix} 5 & -5 & | & 0 \\ -1 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix};$ take  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$ 

For eigenvalue 
$$\lambda_2 = -2$$
:  
 $\begin{bmatrix} -1 & -5 & | & 0 \\ -1 & -5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix};$   
take  
 $\vec{v}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$ 

So the general solution is

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-2x}.$$

Use the initial conditions at x = 0 to find  $c_1, c_2$ :

$$\begin{bmatrix} 1\\-1 \end{bmatrix} = c_1 \begin{bmatrix} 1\\1 \end{bmatrix} + c_2 \begin{bmatrix} -5\\1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1\\-1 \end{bmatrix} = \begin{bmatrix} 1&-5\\1&1 \end{bmatrix} \begin{bmatrix} c_1\\c_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1\\c_2 \end{bmatrix} = \begin{bmatrix} 1&-5\\1&1 \end{bmatrix}^{-1} \begin{bmatrix} 1\\-1 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} c_1\\c_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1&5\\-1&1 \end{bmatrix} \begin{bmatrix} 1\\-1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -4\\-2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2\\-1 \end{bmatrix}.$$
Thus

Thus

$$f_1(x) = -\frac{2}{3}e^{4x} + \frac{5}{3}e^{-2x}$$
 and  $f_2(x) = -\frac{2}{3}e^{4x} - \frac{1}{3}e^{-2x}$ 

6. [avg: 6.1/10] Find the eigenvalues and corresponding eigenvectors of the matrix  $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ .

**Eigenvalues:** find the eigenvalues of A, using properties of determinants.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 1 & 1\\ 1 & \lambda - 2 & 1\\ 1 & 1 & \lambda - 2 \end{bmatrix} = \det \begin{bmatrix} \lambda - 2 & 3 - \lambda & 3 - \lambda \\ 1 & \lambda - 3 & 0\\ 1 & 0 & \lambda - 3 \end{bmatrix}$$
$$= (\lambda - 3)^2 \det \begin{bmatrix} \lambda - 2 & -1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 1 \end{bmatrix} = (\lambda - 3)^2 \det \begin{bmatrix} \lambda - 1 & -1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1 \end{bmatrix}$$
$$= (\lambda - 3)^2 (\lambda - 1 + 1) = \lambda (\lambda - 3)^2$$

Thus the eigenvalues of A are  $\lambda_1 = 3$ , repeated, and  $\lambda_2 = 0$ .

**Eigenvectors:** find the corresponding eigenvectors of A by finding (basic) solutions to the homogeneous system  $(\lambda I - A)\vec{v} = \vec{0}$ .

For eigenvalue  $\underline{\lambda_1 = 3}$ :

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 1 & 1 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}; \text{ take } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

For eigenvalue  $\underline{\lambda_2 = 0}$ :

$$\begin{bmatrix} -2 & 1 & 1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}; \text{ take } \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- 7. [avg: 3.1/10] Three short proofs:
  - (a) [5 marks] Let  $\vec{x}, \vec{y}$  be non-zero vectors in  $\mathbb{R}^n$ . Prove that

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

if and only if  $\vec{x}$  and  $\vec{y}$  are orthogonal.

**Solution:** this is a proof of the Pythagorean Theorem; see figure to the right.

$$\begin{split} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + \|\vec{y}\|^2 \\ \Leftrightarrow \quad (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} \\ \Leftrightarrow \quad \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} \\ \Leftrightarrow \quad \vec{x} \cdot \vec{y} &= 0 \end{split}$$



The Pythagorean Theorem

if and only if  $\vec{x}$  and  $\vec{y}$  are orthogonal.

(b) [2 marks] Let A be an  $n \times n$  matrix. Prove that if  $\lambda = 0$  is an eigenvalue of A then A is not invertible.

**Solution:** let  $\vec{v}$  be an eigenvector of A with eigenvalue  $\lambda = 0$ . By definition,  $\vec{v} \neq \vec{0}$  and

$$A\vec{v} = 0\,\vec{v} = \vec{0}.$$

Thus the homogeneous system with coefficient matrix A has a non-trivial solution, so A is not invertible.

**Or:** if  $\lambda = 0$  is an eigenvalue of A then

$$\det(A - 0I) = 0 \Rightarrow \det(A) = 0,$$

so A is not invertible.

(c) [3 marks] Let A be an  $5 \times 5$  matrix such that  $A^T = -A$ . Prove that A is not invertible.

Solution: use properties of determinants.

$$\det(A) = \det(A^T) = \det(-A) = (-1)^5 \det(A) = -\det(A)$$
$$\Rightarrow \det(A) = 0.$$

so A is not invertible.

8. [10 marks; avg: 7.3/10] Match the matrix on the left with the geometric transformation on the right, by putting the letter beside the matrix into the correct blank on the right. (1 mark for each correct choice)

| (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$   | a reflection in the line $x_2 = x_1$ :                     | (i) |
|---|--|-----|
| (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$   | a projection onto the line $x_2 = x_1$ :                   | (g) |
| (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$   | a rotation of $\pi/2$ counter-clockwise around the origin: | (a) |
| (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  | a projection onto $x_2$ -axis:                             | (j) |
| (e) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$  | a reflection in the $x_2$ -axis:                           | (f) |
| (f) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$   | a projection onto the line $x_2 = -x_1$ :                  | (h) |
|   | a rotation of $\pi/2$ clockwise around the origin:         | (c) |
| $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (h) $\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ | a rotation of $\pi$ counter-clockwise around the origin:   | (d) |
| $\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 $                                    | a reflection in the line $x_2 = -x_1$ :                    | (e) |
| $\begin{bmatrix} 1 & 0 \end{bmatrix}$   | a reflection in the $x_1$ -axis:                           | (b) |
|   |  | ~ / |

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