Solutions to Exam Part 2

Question 1 (3 marks) : True or False?

> If A is an $n \times n$ matrix such that $\text{rank}(A) = \text{rank}(A^2)$, then $A = A^2$.

Answer: False.

Counterexample: pick any invertible matrix A such that $\det(A) \neq 1$, say

$$A = \text{diag}(2, 1, 1, \ldots, 1).$$

Then $\text{rank}(A) = n = \text{rank}(A^2)$, but

$$A^2 = \text{diag}(4, 1, 1, \ldots, 1) \neq A.$$

Or simply say, $\det(A^2) = 4 \neq 2 = \det(A)$, so A can't equal A^2. (Of course, there are many other possible counterexamples.)

Question 2 (3 marks) : True or False?

If A is an $n \times n$ matrix such that $\text{rank}(A) = \text{rank}(A^2)$, then $\text{col}(A) = \text{col}(A^2)$.

Answer: True.

Proof: since $\dim(\text{col}(A)) = \text{rank}(A) = \text{rank}(A^2) = \dim(\text{col}(A^2))$ we know that

$$\dim(\text{col}(A)) = \dim(\text{col}(A^2)).$$

Observe that $\text{col}(A^2) \subseteq \text{col}(A)$:

$$\begin{aligned}
\vec{y} \in \text{col}(A^2) &\Rightarrow \vec{y} = A^2\vec{x}, \text{ for some } \vec{x} \in \mathbb{R}^n \\
&\Rightarrow \vec{y} = A(A\vec{x}) \\
&\Rightarrow \vec{y} = A\vec{z}, \text{ for } \vec{z} = A\vec{x}, \text{ which is in } \mathbb{R}^n \\
&\Rightarrow \vec{y} \in \text{col}(A)
\end{aligned}$$

Then by Theorem 5.2.8, $\text{col}(A^2) = \text{col}(A)$.

Question 3 (3 marks) : True or False?

If A is an $n \times n$ matrix such that $\text{rank}(A) = \text{rank}(A^2)$, then $\text{null}(A) = \text{null}(A^2)$.

Answer: True

Proof: since $\dim(\text{null}(A)) = n - \text{rank}(A) = n - \text{rank}(A^2) = \dim(\text{null}(A^2))$ we know that

$$\dim(\text{null}(A)) = \dim(\text{null}(A^2)).$$

Observe that $\text{null}(A) \subseteq \text{null}(A^2)$:

$$\begin{aligned} \vec{x} \in \text{null}(A) &\Rightarrow A\vec{x} = \vec{0} \\ &\Rightarrow A(A\vec{x}) = A\vec{0} \\ &\Rightarrow A^2\vec{x} = \vec{0} \\ &\Rightarrow \vec{x} \in \text{null}(A^2) \end{aligned}$$

Then by Theorem 5.2.8, $\text{null}(A) = \text{null}(A^2)$.

Question 4 (5 marks) : Let U be a subspace of $\mathbb{R}^n$; let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthonormal basis for U; and let A be the $n \times k$ matrix $[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_k]$. That is, let

$$A = [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_k]\,.$$

Show that for all $\vec{x} \in \mathbb{R}^n$,

$$\mathrm{proj}_U(\vec{x}) = AA^T\,\vec{x}.$$

Solution: using the projection formula, and the fact that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal basis for U, so that $\|\vec{v}_i\| = 1$, we have

$$\mathrm{proj}_U(\vec{x}) = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + \cdots + (\vec{x}\cdot\vec{v}_k)\vec{v}_k.$$

Now use properties of matrix multiplication and properties of dot product:

$$\begin{aligned}
\mathrm{proj}_U(\vec{x}) &= (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + \cdots + (\vec{x}\cdot\vec{v}_k)\vec{v}_k \\
&= (\vec{v}_1\cdot\vec{x})\vec{v}_1 + (\vec{v}_2\cdot\vec{x})\vec{v}_2 + \cdots + (\vec{v}_k\cdot\vec{x})\vec{v}_k \\
&= [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_k]\begin{bmatrix} \vec{v}_1\cdot\vec{x} \\ \vec{v}_2\cdot\vec{x} \\ \vdots \\ \vec{v}_k\cdot\vec{x} \end{bmatrix} \\
&= A\begin{bmatrix} \vec{v}_1^T\vec{x} \\ \vec{v}_2^T\vec{x} \\ \vdots \\ \vec{v}_k^T\vec{x} \end{bmatrix} \\
&= A\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_k^T \end{bmatrix}\vec{x} \\
&= AA^T\,\vec{x}
\end{aligned}$$

Alternate Solution for Question 4: we have $U = \mathrm{col}(A)$. As with the derivation of least squares approximations, there is a $\vec{z} \in \mathbb{R}^k$ such that

$$\mathrm{proj}_U(\vec{x}) = A\,\vec{z} \text{ and } \vec{x} - A\,\vec{z} \in U^\perp.$$

As shown in class, $U^\perp = \mathrm{null}(A^T)$. Thus

$$A^T(\vec{x} - A\vec{z}) = \vec{0} \Leftrightarrow A^T\vec{x} = A^TA\,\vec{z}.$$

Since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal basis,

$$\vec{v}_i^T\vec{v}_j = \vec{v}_i\cdot\vec{v}_j = \left\{\begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{array}\right.,$$

for $1 \leq i, j \leq k$. Thus $A^TA = I_{k\times k}$ and $A^T\,\vec{x} = \vec{z}$. Multiplying both sides by A on the left gives:

$$AA^T\,\vec{x} = A\,\vec{z} = \mathrm{proj}_U(\vec{x}).$$

Question 5 (8 marks) : Let

$$U = \text{span}\left\{\begin{bmatrix} 1\\0\\1\\-1\\1 \end{bmatrix}, \begin{bmatrix} 2\\1\\3\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\2\\1\\1\\1 \end{bmatrix}\right\} \text{ and } \vec{x} = \begin{bmatrix} 1\\-6\\1\\1\\3 \end{bmatrix}.$$

Find vectors $\vec{u}$ and $\vec{v}$ such that

$$\vec{u} \in U, \vec{v} \in U^{\perp} \text{ and } \vec{x} = \vec{u} + \vec{v}.$$

Solution: $\vec{u} = \text{proj}_U(\vec{x})$ and $\vec{v} = \text{proj}_{U^{\perp}}(\vec{x})$. You only have to calculate one directly. To do so you need to find an orthogonal basis for U or an orthogonal basis for $U^{\perp}$.

Easiest Way: let

$$\vec{x}_1 = \begin{bmatrix} 1\\0\\1\\-1\\1 \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} 2\\1\\3\\1\\0 \end{bmatrix}, \ \vec{x}_3 = \begin{bmatrix} -1\\2\\1\\1\\1 \end{bmatrix}.$$

Make use of the fact that $\vec{x}_1 \cdot \vec{x}_3 = 0$. Then an orthogonal basis for U is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ with $\vec{v}_1 = \vec{x}_1, \vec{v}_2 = \vec{x}_3$ and

$$\vec{v}_3 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{x}_2 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{x}_1}{\|\vec{x}_1\|^2}\vec{x}_1 - \frac{\vec{x}_2 \cdot \vec{x}_3}{\|\vec{x}_3\|^2}\vec{x}_3$$

$$= \begin{bmatrix} 2\\1\\3\\1\\0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1\\0\\1\\-1\\1 \end{bmatrix} - \frac{4}{8}\begin{bmatrix} -1\\2\\1\\1\\1 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1\\0\\1\\1\\-1 \end{bmatrix}.$$

So

$$\vec{u} = \text{proj}_U(\vec{x}) = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\vec{x} \cdot \vec{v}_3}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= \frac{4}{4}\begin{bmatrix} 1\\0\\1\\-1\\1 \end{bmatrix} - \frac{8}{8}\begin{bmatrix} -1\\2\\1\\1\\1 \end{bmatrix} + \frac{0}{4}\begin{bmatrix} 1\\0\\1\\1\\-1 \end{bmatrix} = \begin{bmatrix} 1\\0\\1\\-1\\1 \end{bmatrix} - \begin{bmatrix} -1\\2\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 2\\-2\\0\\-2\\0 \end{bmatrix};$$

and consequently,

$$\vec{v} = \vec{x} - \vec{u} = \begin{bmatrix} 1\\-6\\1\\1\\3 \end{bmatrix} - \begin{bmatrix} 2\\-2\\0\\-2\\0 \end{bmatrix} = \begin{bmatrix} -1\\-4\\1\\3\\3 \end{bmatrix}.$$

Direct Approach: directly apply the Gram-Schmidt algorithm to the given basis for U to get an orthogonal basis for U. Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \ \vec{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for U, with $\vec{v}_1 = \vec{x}_1$;

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

and

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{11}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \frac{3}{11}\begin{bmatrix} -5 \\ 6 \\ 1 \\ 1 \\ 5 \end{bmatrix}.$$

So

$$\vec{u} = \mathrm{proj}_U(\vec{x}) = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\vec{x} \cdot \vec{v}_3}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= \frac{4}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{11}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{24}{88}\begin{bmatrix} -5 \\ 6 \\ 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{11}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{11}\begin{bmatrix} -5 \\ 6 \\ 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 0 \end{bmatrix};$$

and as before,

$$\vec{v} = \vec{x} - \vec{u} = \begin{bmatrix} 1 \\ -6 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

Alternate Approach: calculate an orthogonal basis for $U^\perp$. The advantage of this method is that $\dim(U^\perp) = 2$. But first you need to find a basis for $U^\perp$:

$$U^\perp = \text{null}\begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 2 & 1 & 1 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Call these basic solutions $\vec{x}_1$ and $\vec{x}_2$, respectively. Then an orthogonal basis for $U^\perp$ is $\vec{v}_1 = \vec{x}_1$ and

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \\ 3 \end{bmatrix}$$

Consequently

$$\vec{v} = \text{proj}_{U^\perp}(\vec{x}) = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{6}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{24}{24}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \\ 3 \\ 3 \end{bmatrix},$$

and

$$\vec{u} = \vec{x} - \vec{v} = \begin{bmatrix} 1 \\ -6 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -4 \\ 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

Question 6 (8 marks) : Find A^k for k a positive integer if

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 2 & 6 & 4 \\ -6 & 4 & 2 \end{bmatrix}.$$

You may assume the eigenvalues of A are

$$\lambda_1 = 6, \ \lambda_2 = -6, \ \lambda_3 = 9.$$

Solution: since A is symmetric you can orthogonally diagonalize A; that is, find an orthogonal matrix P and a diagonal matrix D such that

$$D = P^T AP \Leftrightarrow A = PDP^T.$$

Then, as we saw in Section 3.3/Week 7, and using the fact that $P^{-1} = P^T$,

$$A^k = PD^k P^T.$$

Step 1: since the eigenvalues of A are all distinct, we only have to find an eigenvector for each eigenvalue, and then divide each eigenvector by its length so that the columns of P are unit vectors. (If you don't divide the eigenvectors by their length, things will still work but calculating P^{-1} will be messier.)

$$E_6(A) = \text{null}(6I - A) = \text{null} \begin{bmatrix} 5 & -2 & 6 \\ -2 & 0 & -4 \\ 6 & -4 & 4 \end{bmatrix} = \text{null} \begin{bmatrix} 5 & -2 & 6 \\ 1 & 0 & 2 \\ 6 & -4 & 4 \end{bmatrix}$$

$$= \text{null} \begin{bmatrix} 0 & -2 & -4 \\ 1 & 0 & 2 \\ 0 & -4 & -8 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}; \text{ take } \vec{v}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

$$E_{-6}(A) = \text{null}(-6I - A) = \text{null} \begin{bmatrix} -7 & -2 & 6 \\ -2 & -12 & -4 \\ 6 & -4 & -8 \end{bmatrix} = \text{null} \begin{bmatrix} -7 & -2 & 6 \\ 1 & 6 & 2 \\ 6 & -4 & -8 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 40 & 20 \\ 1 & 6 & 2 \\ 0 & -40 & -20 \end{bmatrix}$$

$$= \text{null} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}; \text{ take } \vec{v}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

$E_9(A)$: you can continue with the same approach as above; OR since eigenvectors corresponding to distinct eigenvalues of a symmetric matrix must be orthogonal, you can take the cross-product of the previous two eigenvectors to get a third one:

$$\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 6 \end{bmatrix}; \text{ so take } \vec{v}_3 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$$

Step 2: thus we have

$$P = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Note that P is itself symmetric, so $P^{-1} = P^T = P$.

Step 3: compute!

$$\begin{aligned}
A^k &= PD^kP^T = PD^kP \\
&= \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}^k \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 6^k & 0 & 0 \\ 0 & (-6)^k & 0 \\ 0 & 0 & 9^k \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2\cdot 6^k & 2\cdot 6^k & -6^k \\ 2\cdot(-6)^k & -(-6)^k & 2\cdot(-6)^k \\ -9^k & 2\cdot 9^k & 2\cdot 9^k \end{bmatrix} \\
&= \frac{1}{9}\left[\begin{array}{c|c|c} 4\cdot 6^k + 4\cdot(-6)^k + 9^k & 4\cdot 6^k - 2\cdot(-6)^k - 2\cdot 9^k & -2\cdot 6^k + 4\cdot(-6)^k - 2\cdot 9^k \\ 4\cdot 6^k - 2\cdot(-6)^k - 2\cdot 9^k & 4\cdot 6^k + (-6)^k + 4\cdot 9^k & -2\cdot 6^k - 2\cdot(-6)^k + 4\cdot 9^k \\ -2\cdot 6^k + 4\cdot(-6)^k - 2\cdot 9^k & -2\cdot 6^k - 2\cdot(-6)^k + 4\cdot 9^k & 6^k + 4\cdot(-6)^k + 4\cdot 9^k \end{array}\right]
\end{aligned}$$