University of Toronto, Faculty of Applied Science and Engineering

MAT187H1S - Calculus II

Final Exam - April 24, 2015

Examiners: D. Burbulla, S. Cohen, B. Galvão-Sousa, P. Milgram, D. Panchenko, M. Pawliuk, L.-P. Thibault

Time allotted: 150 minutes

No Aids permitted

Total marks: 100

Solutions

Full Name: ______________________________
Last First

Student ID: ______________________________

Email: ______________________________ @mail.utoronto.ca

<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages and a detached **formula sheet**. Make sure you have all of them.
- You can use pages 13–14 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 13–14.

GOOD LUCK!

PART I. No explanation is necessary. **(20 Marks)**

1. (2 marks) Compute the following integral.

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}}\,dx = \underline{\quad \frac{\pi}{4\sqrt{2}} \quad}.$$

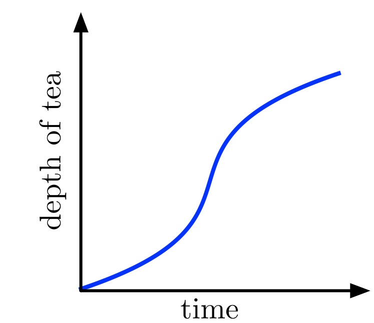
2. (2 marks) When **calculating the integral**

$$\int \frac{x^4+2}{x(x-3)^2(1+x^2)}\,dx,$$

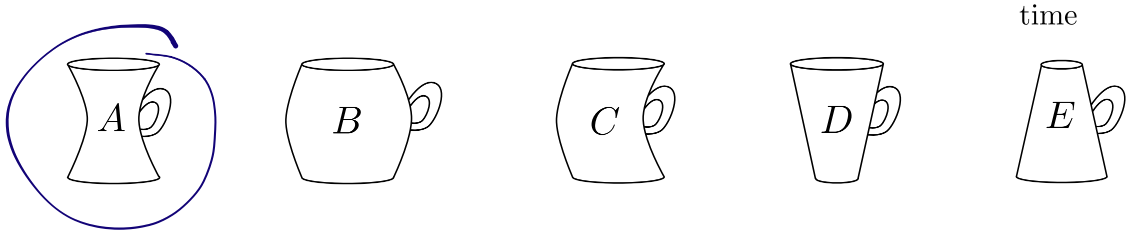
we obtain the sum of the following terms (**circle all that apply**):

(a) $A\ln\lvert x\rvert$	**(d)** $D\ln\lvert x-3\rvert$	**(g)** $G\ln(1+x^2)$	**(j)** $J\arctan(1+x^2)$
(b) $\frac{B}{x}$	**(e)** $\frac{E}{x-3}$	**(h)** $\frac{H}{1+x^2}$	**(k)** $K\arctan(x)$
(c) C	**(f)** $\frac{F}{(x-3)^2}$	**(i)** $\frac{Ix}{1+x^2}$	**(l)** $L\,x\arctan(x)$

3. (2 marks) Tea is poured in a mug at a constant rate (constant volume per unit time). The graph on the right shows the depth of tea in the mug as a function of time.



Which of the mugs below was used? (**circle one option**).



Continued...

4. **(2 marks)** Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. How many terms do we need to add to guarantee that the error of the approximation is smaller than $\frac{1}{2016}$?

$p =$ 2015

5. **(2 marks)** Approximate $f(\theta) = \sin\left(\frac{\theta^2}{3}\right)$ with a 6^{th}-order Taylor polynomial centred at $\theta = 0$.

$p_6(\theta) = \frac{\theta^2}{3} - \frac{\theta^6}{3^3 \cdot 3!}$.

For **questions 6–9**, consider the power series:

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{4n+5}}(x-1)^n$$

6. **(2 marks)** The radius of convergence is

$R =$ 1.

7. **(2 marks)** The interval of convergence is

$x \in$ $(0, 2]$.

8. **(2 marks)** $f^{(2016)}(1) = \frac{2016!}{\sqrt{4 \cdot 2016 + 5}}$.

9. **(2 marks)** Consider the initial-value problem

$$\begin{cases} u'(x) = f(x) \\ u(1) = -2 \end{cases}$$

where $f(x)$ is defined above. Then, the Taylor series for the solution is

$u(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{4n+5}} \frac{(x-1)^{n+1}}{n+1} - 2$.

10. **(2 marks)** (Hard!) Write a **power series** that converges for $x \in [7, 9.5)$.

$\sum_{n=1}^{\infty} \frac{1}{n} \cdot \left(\frac{x - 8.25}{1.25}\right)^n$

Continued...

PART II. Answer the following questions. **Justify** your answers.

11. Air pressure $p(y)$ at altitude y (in metres) is the force per unit area exerted by the weight of the air above, so **(20 Marks)**

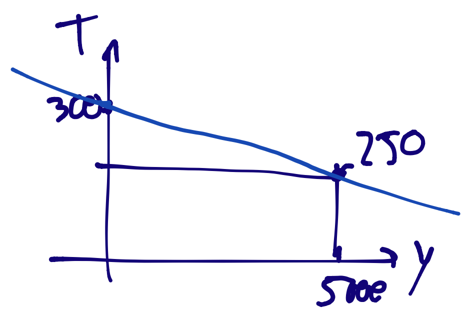
$$p(y) = g \int_y^{30\,000} f(z)\,dz,$$

where $f(y) =$ density of air at altitude y, which is related to the atmospheric pressure:

$$p(y) = Rf(y)T(y),$$

where R is a constant and $T(y)$ is the temperature (in K) at altitude y.

(a) **(5 marks)** The temperature is a linear function of the altitude. Assume that the altitude is measured from sea-level, the temperature at sea-level is 300K and at 5000m is 250K. Find an expression for $T(y)$.



$$T(y) = 300 - \frac{50}{5000}y = 300 - \frac{y}{100} = \frac{30\,000 - y}{100}$$

(b) **(5 marks)** Show that $p(y)$ satisfies the differential equation

$$p'(y) = -\frac{g}{R}\frac{p(y)}{T(y)}.$$

$$p(y) = \frac{g}{R}\int_y^{30000} \frac{p(z)}{T(z)}\,dz = -\frac{g}{R}\int_{30000}^{y} \frac{p(z)}{T(z)}\,dz$$

So taking a derivative:

$$\boxed{p'(y) = -\frac{g}{R}\frac{p(y)}{T(y)}}$$

(c) **(8 marks)** Assume that the air pressure at sea level is 10^5 Pa.

Find a formula for $p(y)$ (it can depend on R and g).

We have the initial condition $p(0) = 10^5$.

This is a separable DE, so

$$\int \frac{1}{p}\,dp = -\frac{g}{R}\int \frac{1}{T(y)}\,dy = -\frac{100g}{R}\int \frac{1}{30000-y}\,dy$$

$$\Leftrightarrow \ln p = \frac{100g}{R}\ln(30000-y) + C$$

$$\Leftrightarrow p = A\, e^{\frac{100g}{R}\ln(30000-y)} = A\,(30000-y)^{\frac{100g}{R}} \qquad (A = e^C)$$

Using the initial condition: $p(0) = A\; 30000^{\frac{100g}{R}} = 10^5$

$$\Rightarrow A = 10^5 \cdot 30000^{-\frac{100g}{R}}$$

$$\boxed{p(y) = 10^5 \cdot 30000^{-\frac{100g}{R}}(30000-y)^{\frac{100g}{R}} = 10^5\left(1 - \frac{y}{30000}\right)^{\frac{100g}{R}}}$$

(d) **(2 marks)** The speed of sound v_s satisfies $v_s(y) = \sqrt{\dfrac{p(y)}{f(y)}}$.

Show that the speed of sound decreases with the altitude.

Using the equation relating p and f:

$$V_s(y) = \sqrt{RT(y)}$$

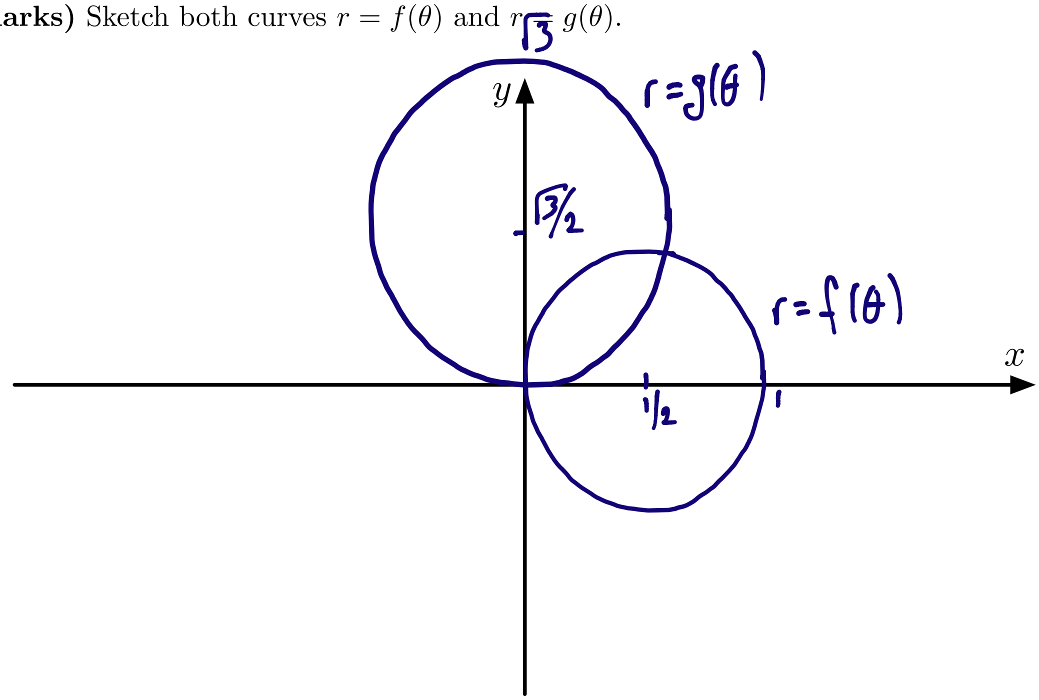
Since $T(y)$ is decreasing with y altitude, then so is $V_s(y)$ the speed of sound.

Continued...

12. Consider the polar curves **(20 Marks)**

$$r = f(\theta) = \cos(\theta) \qquad \text{and} \qquad r = g(\theta) = \sqrt{3}\sin(\theta).$$

(a) **(5 marks)** Sketch both curves $r = f(\theta)$ and $r = g(\theta)$.



(b) **(5 marks)** These two polar curves intersect at two points P and Q. What are the two points of intersection?

They both intersect at $r=0$: $\begin{cases} 0 = f(\pi/2) \\ 0 = g(0) \end{cases}$

To find the other intersection point : $f(\theta) = g(\theta) \iff \cos\theta = \sqrt{3}\sin\theta$

$\iff \tan\theta = \frac{1}{\sqrt{3}}$, so at $\theta = \frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right)$ the curves

intersect : $r = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$:

$$P = \left(\ 0\ ,\ 0\ \right)_{x,y}$$

$$Q = \left(\frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{6}\right) = \frac{3}{4}\ ,\ \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4}\right)_{x,y}$$

(c) (10 marks) Calculate the area of the region inside both $r = f(\theta)$ and $r = g(\theta)$.

Notice that the area is not symmetric about the segment PQ, so

$$\text{Area} = \int_{\pi/6}^{\pi/2} \frac{1}{2}(\cos\theta)^2 \, d\theta + \int_0^{\pi/6} \frac{1}{2}\left(\sqrt{3}\sin\theta\right)^2 d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{1+\cos(2\theta)}{4} \, d\theta + \frac{3}{4}\int_0^{\pi/6} \left[1-\cos(2\theta)\right] d\theta$$

$$= \frac{\pi}{24} + \left[\frac{\sin(2\theta)}{8}\right]_{\pi/6}^{\pi/2} + \frac{3}{4}\frac{\pi}{6} - \frac{3}{8}\left[\sin(2\theta)\right]_0^{\pi/6}$$

$$= \frac{\pi}{24} - \frac{\sin(\pi/3)}{8} + \frac{\pi}{8} - \frac{3}{8}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{7\pi}{24} - \frac{\sqrt{3}}{4}$$

Area $= \dfrac{7\pi}{24} + \dfrac{\sqrt{3}}{4}$

Continued...

13. Consider a particle moving with position **(20 Marks)**

$$\vec{r}(t) = \Big(2\cos(t) \quad , \quad -\cos(t) + \sqrt{3}\sin(t) \quad , \quad -\cos(t) - \sqrt{3}\sin(t)\Big) \qquad \text{for all } t \geqslant 0.$$

(a) **(5 marks)** Show that this particle is always at the same distance from the origin.

$$|\vec{r}(t)|^2 = 4\cos^2(t) + \big(-\cos(t) + \sqrt{3}\sin(t)\big)^2 + \big(-\cos(t) - \sqrt{3}\sin(t)\big)^2$$

$$= \underline{4\cos^2(t)} + \underline{\cos^2(t)} - \cancel{2\sqrt{3}\cos(t)\sin(t)} + \underline{\underline{3\sin^2(t)}}$$
$$+ \underline{\cos^2(t)} + \cancel{2\sqrt{3}\cos(t)\sin(t)} + 3\underline{\underline{\sin^2(t)}}$$

$$= 6\cos^2(t) + 6\sin^2(t) = 6$$

So $|\vec{r}(t)| = \sqrt{6}$ which is constant

(b) **(5 marks)** Show that the acceleration vector is perpendicular to the velocity vector.

$$\vec{v}(t) = \vec{r}\,'(t) = \big(-2\sin(t),\ \sin(t) + \sqrt{3}\cos(t),\ \sin(t) - \sqrt{3}\cos(t)\big)$$
$$\vec{a}(t) = \vec{r}\,''(t) = \big(-2\cos(t),\ \cos(t) - \sqrt{3}\sin(t),\ \cos(t) + \sqrt{3}\sin(t)\big)$$

Then

$$\vec{v}(t)\cdot\vec{a}(t) = 4\sin(t)\cos(t) + \underline{\big(\sin(t) + \sqrt{3}\cos(t)\big)\big(\cos(t) - \sqrt{3}\sin(t)\big)}$$
$$+ \underline{\underline{\big(\sin(t) - \sqrt{3}\cos(t)\big)\big(\cos(t) + \sqrt{3}\sin(t)\big)}}$$

$$= \cancel{4\sin(t)\cos(t)} + \underline{\cancel{\sin(t)\cos(t)} - \cancel{\sqrt{3}\sin^2(t)} + \cancel{\sqrt{3}\cos^2(t)}}$$
$$\cancel{-3\sin(t)\cos(t)} + \underline{\underline{\cancel{\sin(t)\cos(t)} + \cancel{\sqrt{3}\sin^2(t)} - \cancel{\sqrt{3}\cos^2(t)}}}$$
$$\underline{\underline{\cancel{-3\sin(t)\cos(t)}}}$$

$$= 0$$

So the vectors $\vec{v}(t)$ and $\vec{a}(t)$ are perpendicular.

(c) **(5 marks)** Calculate the curvature of the path of the particle.

We have $|\vec{r}\,'(t)|^2 = 4\sin^2(t) + \sin^2(t) + \cancel{2\sqrt{3}\sin(t)\cos(t)} + 3\cos^2(t)$
$+ \sin^2(t) - \cancel{2\sqrt{3}\sin(t)\cos(t)} + 3\cos^2(t)$

$= 6 \quad \Rightarrow \quad |\vec{r}\,'(t)| = \sqrt{6}$

And $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{r}\,'(t)}{\sqrt{6}}$, so $\vec{T}\,'(t) = \dfrac{\vec{r}\,''(t)}{\sqrt{6}} \Rightarrow |\vec{T}\,'(t)| = \dfrac{|\vec{r}\,''(t)|}{\sqrt{6}}$

Since $\vec{r}\,''(t) = -\vec{r}(t)$, so $|\vec{r}\,''(t)| = |\vec{r}(t)| = \sqrt{6}$.

Then $|\vec{T}\,'(t)| = 1$ and

$$\boxed{\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{6}}}$$

(d) **(5 marks)** Find the binormal vector $\vec{B}(t)$. Does this particle moves on a plane?

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\sqrt{6}}$ and $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \dfrac{\vec{r}\,''(t)}{\sqrt{6}}$

So $\vec{B} = \vec{T}\times\vec{N} = \dfrac{1}{6}\,\vec{v}\times\vec{a} = \dfrac{1}{6}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(t) & \sin(t)+\sqrt{3}\cos(t) & \sin(t)-\sqrt{3}\cos(t) \\ -2\cos(t) & \cos(t)-\sqrt{3}\sin(t) & \cos(t)+\sqrt{3}\sin(t) \end{vmatrix}$

$$\vec{B} = \frac{1}{6}\left(2\sqrt{3},\ 2\sqrt{3},\ 2\sqrt{3}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

The vector $\vec{B}$ is constant, so torsion $\tau = 0$, which means that the particle moves in a plane.

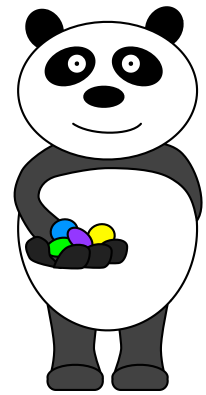
(bonus) **(2 marks)** Based on **(a)** and **(d)**, what is the curve described by this particle?

(You don't need to have solved **(a)** or **(d)** to be able to answer)

The particle moves on the intersection of a plane (from (d)) and a sphere (from (a)), which is a $\boxed{\text{circle.}}$

Continued...

14. After the MAT187 test, the good luck engineering panda arrived home and had a sore arm. It was hard-work giving all that candy and waving good luck to all the students. So he decided to use a spring mechanism for next time! **(20 Marks)**



The panda knows that a spring satisfies the differential equation

$$u''(t) + ku(t) = 0,$$

where $u(t)$ is the displacement from equilibrium (in cm) at time t (in seconds) and k is a constant.

(a) **(5 marks)** The panda initially compresses the spring 4 cm and then lets it go (without initial speed). Find a formula for $u(t)$.

We have the initial conditions $\begin{cases} u(0) = -4 \\ u'(0) = 0 \end{cases}$

To solve for $u(t)$ we look for solutions of the form $u = e^{rt}$. Then the DE implies that

$$r^2 + k = 0 \iff r^2 = -k \iff r = \pm i\sqrt{k}.$$

This means that $u = A\cos(t\sqrt{k}) + B\sin(t\sqrt{k})$.

Using the initial conditions:

$$\begin{cases} -4 = u(0) = A \\ 0 = u'(0) = \sqrt{k}\,B \end{cases} \iff \begin{cases} A = -4 \\ B = 0 \end{cases}$$

The solution is $\boxed{u(t) = -4\cos(\sqrt{k}\,t)}$

Continued...

(b) **(5 marks)** The spring oscillates back and forth. What is the period?

The period of cosine is 2π, so the period of $u(t)$ is P, which satisfies $\sqrt{k}\,P = 2\pi \iff P = \frac{2\pi}{\sqrt{k}}$.

(c) **(5 marks)** The panda needs to deliver candy once every 10 seconds. For that he needs a spring with a specific constant k. Find k.

The panda wants the period to be $10s$:

$$P = \frac{2\pi}{\sqrt{k}} = 10 \iff \boxed{k = \left(\frac{\pi}{5}\right)^2}$$

Continued...

(d) **(5 marks)** The panda could only find an old spring, which satisfies

$$u''(t) + 2u'(t) + \frac{6}{5}u(t) = 0.$$

Find the general solution $u(t)$ that starts with no displacement from equilibrium, but with initial velocity v_0.

The initial conditions are $\begin{cases} u(0) = 0 \\ u'(0) = v_0 \end{cases}$.

Like in (a), we look for solutions of the form $u = e^{rt}$, where $r^2 + 2r + \frac{6}{5} = 0$

Then $r = \frac{-2 \pm \sqrt{4 - 4 \cdot 6/5}}{2} = -1 \pm \sqrt{-1/5} = -1 \pm \frac{i}{\sqrt{5}}$

The solution is $u = A e^{-t} \cos\left(\frac{t}{\sqrt{5}}\right) + B e^{-t} \sin\left(\frac{t}{\sqrt{5}}\right)$

Then $0 = u(0) = A \Rightarrow A = 0$

And $u' = \frac{B e^{-t}}{\sqrt{5}} \cos\left(\frac{t}{\sqrt{5}}\right) - B e^{-t} \sin\left(\frac{t}{\sqrt{5}}\right)$

So $v_0 = u'(0) = \frac{B}{\sqrt{5}} \Rightarrow B = v_0 \sqrt{5}$.

The solution is $\boxed{u(t) = \sqrt{5}\, v_0\, e^{-t} \sin\left(\frac{t}{\sqrt{5}}\right)}$

(bonus) **(2 marks)** Estimate how long it takes for the amplitude of the oscillation $u(t)$ to decay to $\frac{|v_0|}{10\sqrt{5}}$.

The amplitude $|u(t)| = \sqrt{5}|v_0| e^{-t} \left|\sin\left(\frac{t}{\sqrt{5}}\right)\right| \leq \sqrt{5}|v_0| e^{-t}$

We can estimate the time with the left-hand side:

if $\sqrt{5}\cancel{|v_0|} e^{-t} \leq \frac{\cancel{|v_0|}}{10\sqrt{5}} \iff e^{-t} \leq \frac{1}{50}$

$\iff \boxed{t \geq \ln(50)}$

then $|u(t)| \leq \sqrt{5}|v_0| e^{-t} \leq \frac{|v_0|}{10\sqrt{5}}$

$\boxed{\text{Answer: It will take } \ln(50) \text{ seconds}}$.

Continued...