# MAT186H1F - Calculus I - Fall 2015

# Solutions to Term Test 1 - October 13, 2015

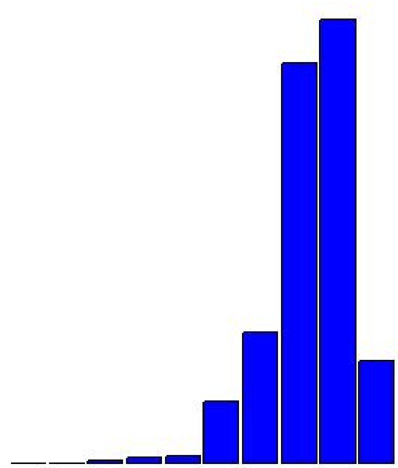
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

**General Comments:**

1. Of the eight questions, Questions 4, 5, 7 and 8 were straightforward, computational problems, with Question 7 the easiest of all. These four questions were done very well.
2. Questions 1 and 3 were graphical in nature; one was very well done, the other poorly done.
3. Question 2, the True or False Question, had some definite "traps" which many students fell into.
4. When stating the Intermediate Value Theorem we will not accept long, circuitous variations, or try to decode vague paraphrases: you must state it precisely–for example, as it is written in the book. Moreover, drawing a picture is not a *statement* of the theorem.

**Breakdown of Results:** 887 students wrote this test. The marks ranged from 22.5% to 100%, and the average was 77.5%. There were four perfect papers. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 8.9% |
| A | 47.2% | 80-89% | 38.3% |
| B | 34.6% | 70-79% | 34.6% |
| C | 11.4% | 60-69% | 11.4% |
| D | 5.3 % | 50-59% | 5.3% |
| F | 1.5% | 40-49% | 0.7% |
| | | 30-39% | 0.6% |
| | | 20-29% | 0.2% |
| | | 10-19% | 0.0% |
| | | 0-9% | 0.0% |

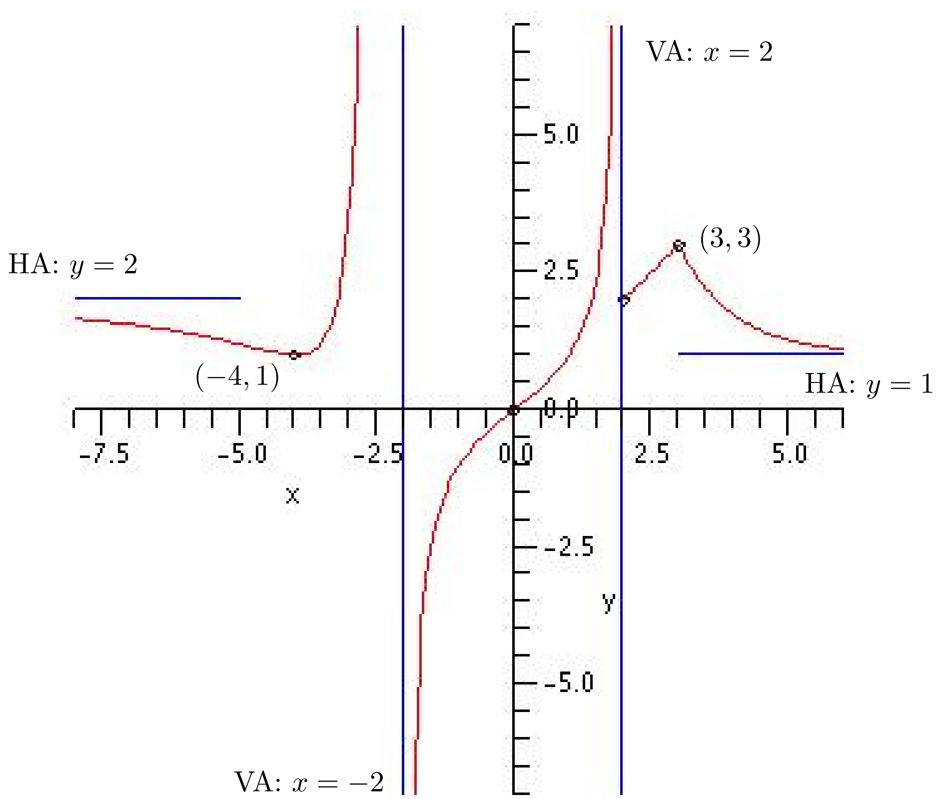


**PART I :** No explanation is necessary for your answers to Question 1 and Question 2.

1. [avg: 8.9/10] Sketch a possible graph of $y = f(x)$ if $f$ is a function with ALL of the following properties:

- $f(x)$ is defined for all $x$, except $x = -2$
- the only solution to $f(x) = 0$ is $x = 0$.
- $f(-4) = 1,\ f(2) = 2$ and $f(3) = 3$
- $f$ is continuous on the intervals $(-\infty, -2), (-2, 2)$ and $[2, \infty)$
- $f$ is differentiable at all $x$ except for $x = -2, x = 2$ and $x = 3$
- $\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$ and $\lim_{x \to 2^-} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = 2$ and $\lim_{x \to \infty} f(x) = 1$
- the only solution to $f'(x) = 0$ is $x = -4$

Clearly label all asymptotes to your graph of $f$. **Solution:** the graph should look something like:



2. [avg: 6.8/10] Decide if the following statements are True or False. Each correct choice is worth 1 mark.

(a) If $f(x)$ is continuous for all $x$ then it is differentiable for all $x$. ○ **True** ⊗ **False**

(b) If a function $f(x)$ is not continuous at $x = a$ then it is not differentiable at $x = a$.

⊗ **True** ○ **False**

(c) For every function $f$ defined for all $x$, if $a < b$ and $f(a) < L < f(b)$ then there is a number $c$ in the open interval $(a, b)$ such that $f(c) = L$. ○ **True** ⊗ **False**

(d) The function $g(x) = \dfrac{x^2 + 5x + 6}{x + 2}$ has a vertical asymptote at $x = -2$. ○ **True** ⊗ **False**

(e) $\displaystyle\lim_{\theta\to\infty} \frac{\sin\theta}{\theta} = 1.$ ○ **True** ⊗ **False**

(f) $\displaystyle\lim_{t\to\infty} t\,\sin(1/t) = 1.$ ⊗ **True** ○ **False**

(g) $\displaystyle\lim_{x\to\infty} x\,\sin x = \infty.$ ○ **True** ⊗ **False**

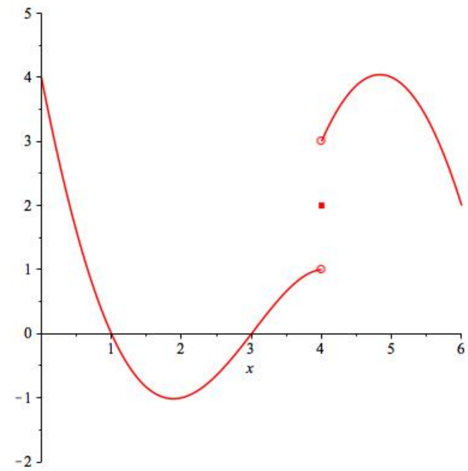
(h) $\dfrac{d}{dx}\left(e^{\pi} + x^2\right) = e^{\pi} + 2x.$ ○ **True** ⊗ **False**

(i) $\displaystyle\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$ ⊗ **True** ○ **False**

(j) $\cos^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{4\pi}{3}.$ ○ **True** ⊗ **False**

**PART II :** Present **complete** solutions to the following questions in the space provided.

3. [avg: 3.3/10] The graph of a function $y = f(x)$ is given to the right. Answer the following questions:




(a) [2 marks] Is $f(x)$ continuous at $x = 4$?

**Soluiton:** No, $f$ has a jump discontinuity at $x = 4$.
More formally, you could observe either one of the following three inequalities to conclude that $f$ is not continuous at $x = 4$:

$$\lim_{x\to 4^-} f(x) = 1 \neq 2 = f(4); \ \lim_{x\to 4^+} f(x) = 3 \neq 2 = f(4); \ \lim_{x\to 4^-} f(x) = 1 \neq 3 = \lim_{x\to 4^+} f(x)$$

(b) [4 marks] Is $f(f(x))$ continuous at $x = 4$?

**Solution:** No. First observe that $f(f(4)) = f(2) = -1$. Then observe that both

$$\lim_{x\to 4^-} f(f(x)) = \lim_{y\to 1^-} f(y) = 0 \neq -1 \text{ and } \lim_{x\to 4^+} f(f(x)) = \lim_{y\to 3^+} f(y) = 0 \neq -1.$$

Either inequality is enough to show $f \circ f$ is not continuous at $x = 4$.
You could also point out that even though

$$\lim_{x\to 4} f(f(x)) = 0,$$

this limit is not equal to $(f \circ f)(4)$. (That is, $f \circ f$ has a removable discontinuity at $x = 4$.)

(c) [4 marks] What is $\lim_{x\to 4} e^{-1/f(f(x))}$ ?

**Solution:** with calculations similar to those of part (b),

$$\lim_{x\to 4} e^{-1/f(f(x))} = \lim_{u\to 0^+} e^{-1/u} = \lim_{v\to\infty} e^{-v} = 0$$

4. [avg: 8.5/10] Suppose $\sin\alpha = \dfrac{3}{5}$ and $\cos\alpha < 0$.

(a) [3 marks] Find the exact value of $\sin(2\alpha)$.

**Solution:** since it is given that $\cos\alpha < 0$, we have

$$\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}.$$

Then

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}.$$

(b) [2 marks] Find the exact value of $\sin(3\alpha)$, given the triple angle formula

$$\sin(3x) = 3\sin x - 4\sin^3 x.$$

**Solution:**

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha = 3\left(\frac{3}{5}\right) - 4\left(\frac{3}{5}\right)^3 = \frac{9}{5} - \frac{108}{125} = \frac{117}{125}.$$

(c) [5 marks] By appropriately differentiating the above triple angle formula for $\sin(3x)$, find a triple angle formula for $\cos(3x)$, and then use it to find the exact value of $\cos(3\alpha)$.

**Solution:**

$$\begin{aligned}
\frac{d\sin(3x)}{dx} &= \frac{d(3\sin x - 4\sin^3 x)}{dx} \\
\Rightarrow 3\cos(3x) &= 3\cos x - 12\sin^2 x\,\cos x \\
\Rightarrow \cos(3x) &= \cos x - 4\sin^2 x\,\cos x \\
\text{(optionally)}\ \Rightarrow \cos(3x) &= -3\cos x + 4\cos^3 x
\end{aligned}$$

Then

$$\cos(3\alpha) = \cos\alpha - 4\sin^2\alpha\,\cos\alpha = -\frac{4}{5} - 4\left(\frac{3}{5}\right)^2\left(-\frac{4}{5}\right) = \frac{44}{125}$$

5. [avg: 8.5/10] Find the following limits.

(a) [5 marks] $\lim_{x\to 2} \frac{x^2-4}{x^3-2x^2+x-2}$

**Solution:** substituting $x = 2$ gives a limit in the 0/0 form, so try factoring.

$$\lim_{x\to 2} \frac{x^2-4}{x^3-2x^2+x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+1)} = \lim_{x\to 2} \frac{x+2}{x^2+1} = \frac{4}{5}.$$

(b) [5 marks] $\lim_{x\to 0} \frac{a+\sqrt{a^2-x^2}}{x^2}$, if $a < 0$.

**Solution:** since $\sqrt{a^2} = |a| = -a$, if $a < 0$, this limit is in the 0/0 form. So try rationalizing.

$$\begin{aligned}
& \lim_{x\to 0} \frac{a+\sqrt{a^2-x^2}}{x^2} \\
= & \lim_{x\to 0} \frac{(a+\sqrt{a^2-x^2})(a-\sqrt{a^2-x^2})}{x^2(a-\sqrt{a^2-x^2})} \\
= & \lim_{x\to 0} \frac{a^2-(a^2-x^2)}{x^2(a-\sqrt{a^2-x^2})} \\
= & \lim_{x\to 0} \frac{x^2}{x^2(a-\sqrt{a^2-x^2})} \\
= & \lim_{x\to 0} \frac{1}{a-\sqrt{a^2-x^2}} \\
= & \frac{1}{a-\sqrt{a^2}} \\
= & \frac{1}{a-(-a)}, \text{ since } a < 0 \\
= & \frac{1}{2a}
\end{aligned}$$

6. [avg: 7.6/10]

6.(a) [4 marks; 2 marks for each part] Use the Squeeze Theorem to find the following limits:

(i) $\lim_{x\to 0} f(x)$, if $-x^2 + \sin x \le f(x) \le x^2 + \sin x$.

**Solution:** both

$$\lim_{x\to 0}(-x^2 + \sin x) = 0$$

and

$$\lim_{x\to 0}(x^2 + \sin x) = 0;$$

so by the Squeeze Theorem,

$$\lim_{x\to 0} f(x) = 0$$

as well.

(ii) $\lim_{x\to -\infty} f(x)$, if $f(x) = e^x \cos(x^3 + 7)$.

**Solution:** since $-1 \le \cos(x^3 + 7) \le 1$, we have

$$-e^x \le f(x) \le e^x.$$

Both

$$\lim_{x\to -\infty} -e^x = 0 \text{ and } \lim_{x\to -\infty} e^x = 0,$$

so

$$\lim_{x\to -\infty} f(x) = 0$$

as well, by the Squeeze Theorem.

6.(b) [6 marks] State the Intermediate Value Theorem and use it to explain why the equation

$$2x + \sin x - \frac{1}{6} = 0$$

has at least one solution.

**Solution:** the Intermediate Value Theorem states:

> Let $f$ be a continuous function on the closed interval $[a, b]$ and suppose $K$ is a value between $f(a)$ and $f(b)$. Then there is a number $c$ in the open interval $(a, b)$ such that $f(c) = K$.

For the second part of the question, let $f(x) = 2x + \sin x - 1/6$, which is continuous for all $x$. Since

$$f(0) = -\frac{1}{6} < 0 \text{ and } f(1) = 2 + \sin 1 - \frac{1}{6} > 0,$$

there is, by the Intermediate Value Theorem, a number $c$ in the interval $(0, 1)$ such that

$$f(c) = 0 \Leftrightarrow 2c + \sin c - \frac{1}{6} = 0.$$

7. [avg: 9.6/10] Suppose a stone is thrown vertically upward from the edge of a cliff with an initial velocity of 20 m/s from a height of 25 m above the ground. Assume the height $s$ (in meters) of the stone above the ground $t$ seconds after it is thrown is $s = -5\,t^2 + 20\,t + 25$.

(a) [2 marks] Determine the velocity $v$ of the stone after $t$ seconds.

**Solution:** in meters per second,

$$v = \frac{ds}{dt} = -10t + 20.$$

(b) [2 marks] When does the stone reach its highest point?

**Solution:** at the highest point, the stone is momentarily at rest.

$$v = 0 \Leftrightarrow -10t + 20 = 0 \Leftrightarrow t = 2.$$

So the stone reaches its highest point at $t = 2$ seconds.

(c) [2 marks] What is the height of the stone at the highest point?

**Solution:** at $t = 2$ the height of the stone is

$$s = -5(2^2) + 20\,2 + 25 = 45$$

meters above the ground.

(d) [2 marks] When does the stone strike the ground?

**Solution:** let $s = 0$ and solve for $t$;

$$-5\,t^2 + 20\,t + 25 \Leftrightarrow t^2 - 4t - 5 = 0 \Leftrightarrow (t+1)(t-5) = 0 \Leftrightarrow t = -1 \text{ or } t = 5.$$

So the stone hits the ground at $t = 5$ seconds.

(e) [2 marks] With what velocity does the stone strike the ground?

**Solution:** at $t = 5$,

$$v = -10(5) + 20 = -30$$

meters per second.

8. [avg: 8.6/10] For each of the graphs determined by the following equations, find the slope of the tangent line to the graph at the point $(x, y) = (1, -1/2)$.

(a) [5 marks] $y = \dfrac{\cos(\pi\sqrt{x})}{(1+x^2)}$.

**Solution:** cross multiply and then use implicit differentiation:

$$y\,(1+x^2) = \cos(\pi\sqrt{x}) \Rightarrow \frac{dy}{dx}\,(1+x^2) + 2x\,y = -\frac{\pi\sin(\pi\sqrt{x})}{2\sqrt{x}}.$$

Substitute $(x, y) = (1, -1/2)$ and solve for $dy/ds$:

$$2\frac{dy}{dx} + 2(1)\left(-\frac{1}{2}\right) = 0 \Leftrightarrow \frac{dy}{dx} = \frac{1}{2}.$$

**Alteranate Solution:** use the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\pi\sin(\pi\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)(1+x^2) - 2x\,\cos(\pi\sqrt{x})}{(1+x^2)^2} \\ \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} &= \frac{0 - 2\,\cos\pi}{2^2} = \frac{1}{2},\ \text{as before.} \end{aligned}$$

(b) [5 marks] $4\,x^2\,y + e^{x-1} = 16\,y^3 + 1$.

**Solution:** use implicit differentiation:

$$8x\,y + 4x^2\frac{dy}{dx} + e^{x-1} = 48y^2\,\frac{dy}{dx}.$$

Substitute $(x, y) = (1, -1/2)$ and solve for $dy/dx$:

$$8\left(-\frac{1}{2}\right) + 4\frac{dy}{dx} + e^0 = 48\left(\frac{1}{4}\right)\frac{dy}{dx} \Leftrightarrow 4\frac{dy}{dx} - 3 = 12\frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = -\frac{3}{8}.$$

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