University of Toronto
Solutions to **MAT186H1F TERM TEST**
of **Tuesday, October 16, 2012**
Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all questions. Present your solutions in the space provided; use the backs of the pages if you need more space. Do not use L'Hopital's rule on this test. The value for each question is indicated in parantheses beside the question number. **Total Marks: 60**

General Comments about the Test:

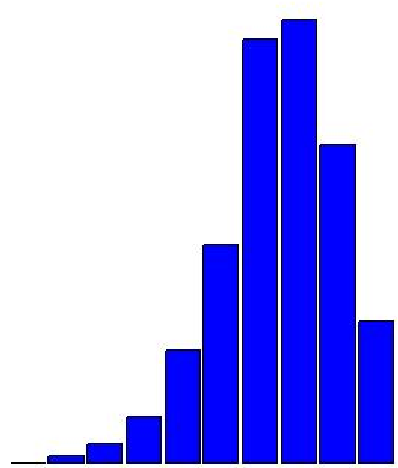
- In a written test you must explain what you are doing to get full credit. The answer by itself is worth very little if you don't explain how you got it. Moreover, you can't just plop down formulas and expect the marker to figure out what you are doing; you are supposed to make it clear what you are doing.
- Questions 1, 2, 3, 4, 5(a), 7, 8 and 9 are all considered straightforward, routine questions. Everybody should have aced these. That's 50 out of the 60 marks on this test.
- Only Questions 5(b) and 6 are considered difficult questions.
- You must refer to the basic trig limit,

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

  to get full marks in the two parts of Question 5.
- Many students still have no clue how to use = or ⇒ correctly!

Breakdown of Results: 523 students wrote this test. The marks ranged from 18.3% to 100%, and the average was 68.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.2%
A	26.6%	80-89%	18.4%
B	25.6%	70-79%	25.6%
C	24.5 %	60-69%	24.5%
D	12.6%	50-59%	12.6%
F	10.7%	40-49%	6.5%
		30-39%	2.7%
		20-29%	1.1%
		10-19%	0.4 %
		0-9%	0.0%



1. [7 marks] Find $\frac{dy}{dx}$ if

(a) [3 marks] $y = \ln(1 + \sin^2 x)$.

Solution: use the chain rule.

$$\frac{dy}{dx} = \frac{1}{1+\sin^2 x}\frac{d(1+\sin^2 x)}{dx} = \frac{2\sin x\cos x}{1+\sin^2 x}$$

(b) [4 marks] $y = e^{\cos x}\sqrt{x^2+1}$.

Solution: use the product rule:

$$\frac{dy}{dx} = e^{\cos x}\frac{2x}{2\sqrt{x^2+1}} - \sin x\, e^{\cos x}\sqrt{x^2+1} = \frac{xe^{\cos x}}{\sqrt{x^2+1}} - \sin x\, e^{\cos x}\sqrt{x^2+1}$$

Or use logarithmic differentiation:

$$\begin{aligned}
\ln y = \cos x + \frac{1}{2}\ln(x^2+1) \quad &\Rightarrow \quad \frac{y'}{y} = -\sin x + \frac{1}{2}\frac{2x}{x^2+1} \\
&\Rightarrow \quad \frac{dy}{dx} = y\left(-\sin x + \frac{x}{x^2+1}\right) \\
&\Rightarrow \quad \frac{dy}{dx} = -\sin x\, e^{\cos x}\sqrt{x^2+1} + \frac{xe^{\cos x}}{\sqrt{x^2+1}}
\end{aligned}$$

2. [7 marks] Given that $\theta = \cos^{-1}\left(-\frac{3}{4}\right)$ find the *exact* values of the following:

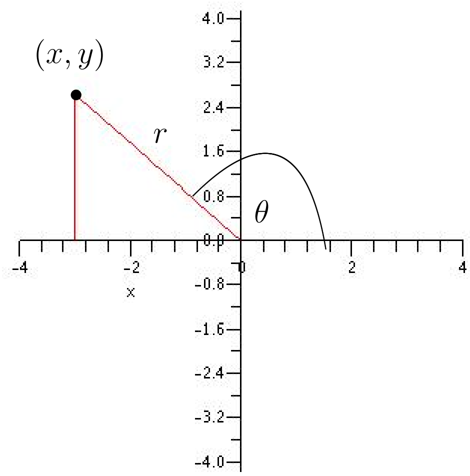
(a) [3 marks] $\tan\theta$

Solution: Note θ is in the second quadrant, by definition of the inverse cosine function. In the triangle to the right,

$$(x, y) = (-3, \sqrt{7}) \text{ and } r = 4.$$

So

$$\tan\theta = \frac{y}{x} = -\frac{\sqrt{7}}{3}.$$



Alternate Solution:

$$\tan^2\theta = \sec^2\theta - 1 = \left(-\frac{4}{3}\right)^2 - 1 = \frac{7}{9} \Rightarrow \tan\theta = -\frac{\sqrt{7}}{3},$$

since θ is in the second quadrant.

(b) [4 marks] $\sin(2\theta)$

Solution: use the appropriate double angle formula.

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{7}}{4}\right)\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8},$$

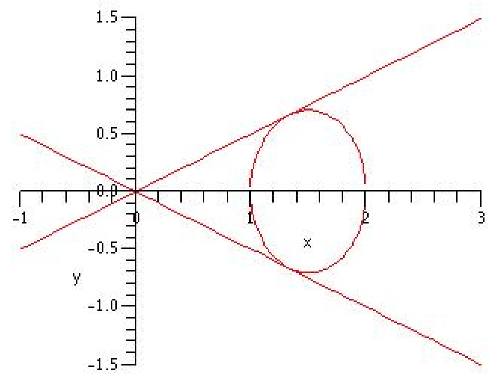
where we calculated $\sin\theta$ from the triangle in part (a). Alternately:

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}.$$

3. [6 marks] Find equations for two lines through the origin that are tangent to the ellipse with equation

$$2x^2 - 6x + y^2 + 4 = 0.$$

The graph of the ellipse is shown to the right.



Solution: differentiate implicitly to find dy/dx:

$$2x^2 - 6x + y^2 + 4 = 0 \Rightarrow 4x - 6 + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3-2x}{y}.$$

Since the tangent line is to go through the origin it must have equation $y = mx$. Let the point of contact of this line and the ellipse be $(x, y) = (a, b)$. Then (1):

$$2a^2 - 6a + b^2 + 4 = 0 \Leftrightarrow b^2 = 6a - 2a^2 - 4.$$

Now calculate the slope of the tangent line in two ways:

$$m = \frac{b}{a} \text{ and } m = \frac{3-2a}{b}.$$

So (2):

$$\frac{b}{a} = \frac{3-2a}{b} \Leftrightarrow b^2 = 3a - 2a^2.$$

Subtracting equation (2) from equation (1) gives

$$3a - 4 = 0 \Leftrightarrow a = \frac{4}{3}.$$

From equation (1),

$$b^2 = 8 - \frac{32}{9} - 4 = \frac{4}{9} \Rightarrow b = \pm\frac{2}{3}.$$

Then

$$m = \frac{b}{a} = \frac{\pm\frac{2}{3}}{\frac{4}{3}} = \pm\frac{1}{2};$$

so the tangent lines have equations

$$y = \pm\frac{x}{2}.$$

4. [7 marks] Find the following limits:

(a) [3 marks] $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 5x + 4}$

Solution: factor the numerator and the denominator.

$$\begin{aligned}
\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 5x + 4} &= \lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)(x-4)} \\
&= \lim_{x \to 1} \frac{x+3}{x-4} \\
&= -\frac{4}{3}
\end{aligned}$$

(b) [4 marks] $\lim_{x \to 0} \frac{\sqrt{4x+9} - 3}{x}$

Solution: rationalize the numerator.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{4x+9} - 3}{x} &= \lim_{x \to 0} \frac{(\sqrt{4x+9} - 3)(\sqrt{4x+9} + 3)}{x\,(\sqrt{4x+9} + 3)} \\
&= \lim_{x \to 0} \frac{4x + 9 - 9}{x\,(\sqrt{4x+9} + 3)} \\
&= \lim_{x \to 0} \frac{4}{\sqrt{4x+9} + 3} \\
&= \lim_{x \to 0} \frac{4}{\sqrt{9} + 3} \\
&= \frac{2}{3}
\end{aligned}$$

5. [7 marks] Find the following limits:

(a) [3 marks] $\lim_{x\to\infty} x \sin\left(\frac{4}{x}\right)$.

Solution: this limit is in the $\infty \cdot 0$ indeterminate form. To reduce this limit to an application of the basic trig limit,

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

let $t = 4/x$:

$$\lim_{x\to\infty} x \sin\left(\frac{4}{x}\right) = 4 \lim_{t\to 0^+} \frac{\sin t}{t} = 4 \cdot 1 = 4.$$

(b) [4 marks] $\lim_{x\to 2} \frac{\sin(\pi x)}{x-2}$

Solution: this limit is in the $0/0$ indeterminate form. To reduce this limit to an application of the basic trig limit, let $h = x - 2$. Then $x = 2 + h$ and

$$\begin{aligned}
\lim_{x\to 2} \frac{\sin(\pi x)}{x-2} &= \lim_{h\to 0} \frac{\sin \pi(h+2)}{h} \\
&= \lim_{h\to 0} \frac{\sin(\pi h + 2\pi)}{h} \\
&= \lim_{h\to 0} \frac{\sin \pi h \, \cos 2\pi + \cos \pi h \, \sin 2\pi}{h} \\
&= \lim_{h\to 0} \frac{\sin \pi h}{h} \\
&= \pi \lim_{h\to 0} \frac{\sin(\pi h)}{\pi h} \\
&= \pi \cdot 1 = \pi,
\end{aligned}$$

making use of the basic trig limit again, with $t = \pi h$.

6. [6 marks] Determine if each of the following functions is continuous at $x = 0$. Justify your answer.

(a) [3 marks] $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$

Solution: for $x \neq 0$,

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0}(x^2),$$

the Squeezing Theorem (or Squeeze Law) implies that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

But $f(0) = 1 \neq 0$, so the function f is not continuous at $x = 0$.

(b) [3 marks] $f(x) = \begin{cases} \sec^{-1}\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ \frac{\pi}{2} & \text{if} \quad x = 0 \end{cases}$

Solution: use $\sec^{-1} y = \cos^{-1}(1/y)$, for $y \neq 0$. Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sec^{-1}\left(\frac{1}{x}\right) = \lim_{x \to 0} \cos^{-1} x = \cos^{-1} 0 = \frac{\pi}{2}.$$

But $f(0) = \pi/2$, so the funciton f is continuous at $x = 0$.

Alternately:

$$\lim_{x \to 0^+} \sec^{-1}\left(\frac{1}{x}\right) = \lim_{h \to \infty} \sec^{-1} h = \frac{\pi}{2}$$

and

$$\lim_{x \to 0^-} \sec^{-1}\left(\frac{1}{x}\right) = \lim_{h \to -\infty} \sec^{-1} h = \frac{\pi}{2};$$

so as before

$$\lim_{x \to 0} f(x) = \frac{\pi}{2}.$$

7.(a) [3 marks] State the Intermediate Value Theorem.

Solution: quoting Theorem 1.5.7 from page 115 of the text book:

> If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$ then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.

7.(b) [3 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 - x^2 - 2 = 0$ has at least one solution in the interval $[1, 2]$.

Solution: let $f(x) = x^3 - x^2 - 2$, which is a polynomial function and so is continuous for all x. Now consider $f(x)$ on the closed interval $[1, 2]$:

$$f(1) = -2 < 0 \text{ and } f(2) = 2 > 0.$$

Let $k = 0$. By the Intermediate Value Theorem there is a number $x \in [1, 2]$ such that

$$f(x) = k \Leftrightarrow x^3 - x^2 - 2 = 0.$$

8. [7 marks] Find both $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = (1, 1)$ if $\quad x^2 + y^2 = 6xy - 4.$

Solution: differentiate implicitly.

$$2x + 2y\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

At $(x, y) = (1, 1)$, this becomes

$$2 + 2\frac{dy}{dx} = 6 + 6\frac{dy}{dx} \Leftrightarrow -4 = 4\frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = -1.$$

To find $\dfrac{d^2y}{dx^2}$, differentiate implicitly once more:

$$2 + 2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 6\frac{dy}{dx} + 6\frac{dy}{dx} + 6x\frac{d^2y}{dx^2}.$$

At $(x, y) = (1, 1)$, $\dfrac{dy}{dx} = -1$, so

$$2 + 2(-1)^2 + 2(1)\frac{d^2y}{dx^2} = 12(-1) + 6(1)\frac{d^2y}{dx^2} \Leftrightarrow 16 = 4\frac{d^2y}{dx^2} \Leftrightarrow \frac{d^2y}{dx^2} = 4.$$

9. [7 marks] Let $f(x) = \dfrac{3-x}{1-x}$.

(a) [3 marks] Verify that f is its own inverse.

Solution: Method I. Verify that $(f \circ f)(x) = x$:

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{3-x}{1-x}\right) = \frac{3 - \frac{3-x}{1-x}}{1 - \frac{3-x}{1-x}} = \frac{3 - 3x - (3-x)}{1 - x - (3-x)} = \frac{-2x}{-2} = x.$$

Method II. Find the formula for f^{-1} by interchanging x and y in the formula for f:

$$x = \frac{3-y}{1-y} \Leftrightarrow x - xy = 3 - y \Leftrightarrow y(1-x) = 3 - x \Leftrightarrow y = \frac{3-x}{1-x}.$$

So $f^{-1}(x) = f(x)$.

(b) [4 marks] Plot the graphs of $y = f(x)$ and $y = x$ in the same plane, indicating the discontinuities of f and any horizontal asymptotes to the graph of f. (NB. Don't worry about critical points; this function doesn't have any.)

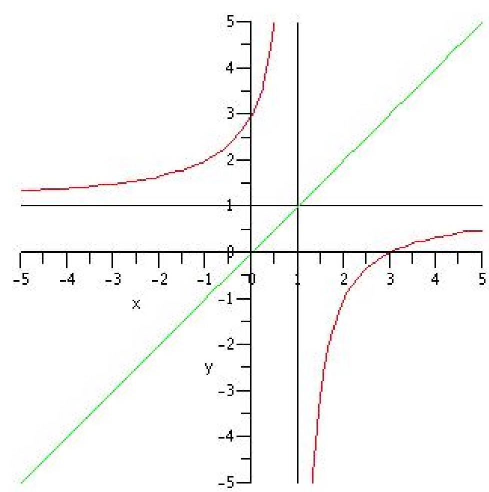
Solution: f has an infinite discontinuity at $x = 1$ since

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{3-x}{1-x} = +\infty$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{3-x}{1-x} = -\infty.$$

This is indicated by the vertical asymptote $x = 1$ to the graph of $y = f(x)$, at right.



There is also a horizontal asymptote $y = 1$ to the graph of $y = f(x)$ since

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{3-x}{1-x} = \lim_{x \to \pm\infty} \frac{3/x - 1}{1/x - 1} = \frac{-1}{-1} = 1.$$

NB: to get full marks the graph of $y = f(x)$, shown in red, must be symmetric with respect to the line $y = x$, shown in green.