University of Toronto

SOLUTIONS to MAT186H1F TERM TEST 1

of **Tuesday, October 7, 2008**

Duration: 90 minutes

TOTAL MARKS: 60

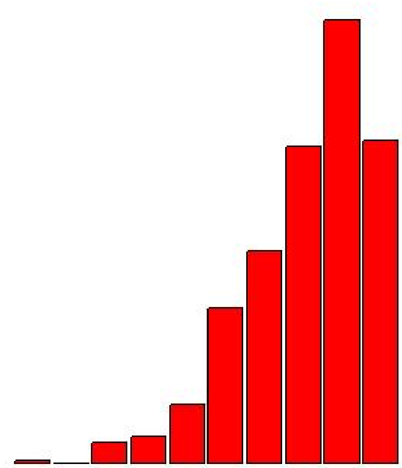
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Many of the questions were based on homework problems, sometimes verbatim.
- In Question #1 decimal approximations are not acceptable.
- In Question #2(b) you must state that the function $f(x) = x^4 + 2x - 1$ is continuous; otherwise the Intermediate Value Property cannot be applied.
- The point to Question #4(a) is to manipulate the given expressions until the basic trig limit can be used.
- Question #5(b) is the hardest part of the test.
- In Question #6 there are two critical points, one for which $f'(x) = 0$, and one for which $f'(x)$ is undefined. This must be clearly indicated to get full marks.
- In Question #7 the point $(1, 5)$ is *not* on the graph of $f(x) = x^3$, so calculating $f'(1)$ and using that as the slope of the required tangent line is a conceptual blunder of the worst kind.
- Far too many students abuse mathematical notation, for which they lost marks.

Breakdown of Results: 451 students wrote this test. The marks ranged from 6.7% to 100%, and the average was 75.1%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	20.9%
A	48.8 %	80-89%	27.9%
B	20.4%	70-79%	20.4%
C	13.7%	60-69%	13.7%
D	10.0%	50-59%	10.0%
F	7.1%	40-49%	3.8%
		30-39%	1.8%
		20-29%	1.3%
		10-19%	0.0%
		0-9%	0.2%



1. [9 marks] Suppose $\cos x = -\dfrac{2}{7}$ and $\sin x > 0$. Find the exact values of the following:

   (a) [2 marks] $\sin x$

   Solution:

$$\begin{aligned} \sin x &= \sqrt{1-\cos^2 x} \\ &= \sqrt{1-\left(-\frac{2}{7}\right)^2} \\ &= \sqrt{1-\left(\frac{4}{49}\right)} \\ &= \sqrt{\frac{45}{49}} \\ &= \frac{3\sqrt{5}}{7} \end{aligned}$$

   (b) [3 marks] $\sin(2x)$

   Solution:

$$\begin{aligned} \sin(2x) &= 2\sin x\cos x \\ &= 2\left(\frac{3\sqrt{5}}{7}\right)\left(-\frac{2}{7}\right) \\ &= -\frac{12}{49}\sqrt{5} \end{aligned}$$

   (c) [4 marks] $\cos\left(x-\dfrac{\pi}{3}\right)$

   Solution:

$$\begin{aligned} \cos\left(x-\frac{\pi}{3}\right) &= \cos x\,\cos\frac{\pi}{3}+\sin x\,\sin\frac{\pi}{3} \\ &= \left(-\frac{2}{7}\right)\left(\frac{1}{2}\right)+\left(\frac{3\sqrt{5}}{7}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{3\sqrt{15}-2}{14} \end{aligned}$$

2(a) [4 marks] Find $g'(1)$ if $g(t) = \sqrt{t + \sqrt{t}}$.

Solution: Use the chain rule.

$$\begin{aligned} g'(t) &= \frac{1}{2}\frac{1}{\sqrt{t+\sqrt{t}}}\left(1 + \frac{1}{2}\frac{1}{\sqrt{t}}\right) \\ \Rightarrow g'(1) &= \frac{1}{2}\frac{1}{\sqrt{1+1}}\left(1+\frac{1}{2}\right) = \frac{3}{4\sqrt{2}} \text{ or } \frac{3\sqrt{2}}{8} \end{aligned}$$

2(b) [5 marks] Use the Intermediate Value Property to explain why the equation $x^4+2x-1=0$ has at least two solutions in the interval $[-2, 2]$. (Be succinct; be precise; do not beat around the bush.)

Solution: Let $f(x) = x^4 + 2x - 1$, which is a continuous function for all x. Observe that

$$f(-2) = 16 - 4 - 1 = 11 > 0 \text{ and } f(0) = -1 < 0.$$

So by the Intermediate Value Property, there is a number $c_1 \in (-2, 0)$ such that

$$f(c_1) = 0.$$

Similarly,

$$f(0) = -1 < 0 \text{ and } f(2) = 16 + 4 - 1 = 19 > 0;$$

so by the Intermediate Value Property, there is a number $c_2 \in (0, 2)$ such that

$$f(c_2) = 0.$$

Thus the equation $f(x) = 0$ has at least two solutions in the interval $[-2, 2]$.

3. [8 marks] Find the following limits.

(a) [4 marks] $\lim_{x\to 1} \frac{x^3-1}{x^2+2x-3}$

Solution: Factor and simplify.

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-1}{x^2+2x-3} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x+3)(x-1)} \\ &= \lim_{x\to 1} \frac{x^2+x+1}{x+3} \\ &= \frac{3}{4}\end{aligned}$$

(b) [4 marks] $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

Solution: Rationalize and simplify.

$$\begin{aligned}\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 0} \left(\frac{x-9}{\sqrt{x}-3}\right)\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) \\ &= \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} \\ &= \lim_{x\to 9} \sqrt{x}+3 \\ &= \sqrt{9}+3 \\ &= 6\end{aligned}$$

4. [9 marks] Find the following limits.

(a) [5 marks] $\lim_{z\to 0} \frac{\tan(5z)}{\sin(3z)}$

Solution: Make use of the basic trig limit $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

$$\begin{aligned}
\lim_{z\to 0} \frac{\tan(5z)}{\sin(3z)} &= \lim_{z\to 0} \left(\frac{1}{\cos(5z)} \frac{\sin(5z)}{z} \frac{z}{\sin(3z)} \right) \\
&= \frac{1}{1} \cdot \frac{5}{3} \cdot \lim_{z\to 0} \frac{\sin(5z)}{(5z)} \cdot \lim_{z\to 0} \frac{(3z)}{\sin(3z)} \\
&= \frac{5}{3} \cdot 1 \cdot \frac{1}{1} \\
&= \frac{5}{3}
\end{aligned}$$

(b) [4 marks] $\lim_{x\to 0} x^2 \sin\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)$

Solution: Use the Squeeze Law.

$$\begin{aligned}
x \neq 0 \quad &\Rightarrow \quad -1 \le \sin\left(1 + \frac{1}{x} + \frac{1}{x^2}\right) \le 1 \\
&\Rightarrow \quad -x^2 \le x^2 \sin\left(1 + \frac{1}{x} + \frac{1}{x^2}\right) \le x^2
\end{aligned}$$

Both

$$\lim_{x\to 0}(-x^2) = 0 \text{ and } \lim_{x\to 0} x^2 = 0,$$

so by the Squeeze Law,

$$\lim_{x\to 0} x^2 \sin\left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 0$$

as well.

5. [9 marks] Let $f(x) = \dfrac{1}{1-|x|}$.

(a) [6 marks] Find both points where the function is not defined, and at each such point $x = a$, calculate both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, or explain why they do not exist.

Solution: $f(x)$ is undefined if and only if $1 - |x| = 0 \Leftrightarrow x = \pm 1$.
At $x = 1$,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{1-x} = -\infty, \text{ since } 1 - x \to 0^-;$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{1-x} = +\infty, \text{ since } 1 - x \to 0^+.$$

At $x = -1$, use $|x| = -x$, since $x < 0$. Then

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{1}{1+x} = -\infty, \text{ since } x + 1 \to 0^-$$

and

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{1}{1+x} = +\infty, \text{ since } x + 1 \to 0^+.$$

(b) [3 marks] Is f differentiable at $x = 0$?

Solution: NO. Use the definition of $f'(0)$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1-|h|} - 1}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h(1-|h|)} \end{aligned}$$

This limit doesn't exist, since the two one-sided limits at $h = 0$ are different:

$$\lim_{h\to 0^+} \frac{|h|}{h(1-|h|)} = \lim_{h\to 0^+} \frac{h}{h(1-h)} = \lim_{h\to 0^+} \frac{1}{1-h} = 1$$

and

$$\lim_{h\to 0^-} \frac{|h|}{h(1-|h|)} = \lim_{h\to 0^-} \frac{-h}{h(1+h)} = \lim_{h\to 0^+} \frac{-1}{1+h} = -1.$$

6. [8 marks] Find the maximum and minimum values of the function

$$f(x) = 5x^{2/3} - x^{5/3}$$

on the closed interval $[-1, 4]$.

Solution: This is Example 6 on page 151 of the text book.

$$\begin{aligned} f'(x) &= \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} \\ &= \frac{5}{3}\left(\frac{2-x}{x^{1/3}}\right) \end{aligned}$$

Critical points:

$$f'(x) = 0 \Rightarrow 2 - x = 0 \Rightarrow x = 2;$$

$$f'(x) \text{ is undefined when } x = 0.$$

The endpoints are $x = -1$ or $x = 4$. Compare:

- $f(-1) = 5 - (-1) = 6$
- $f(0) = 0$
- $f(2) = 2^{2/3}(5 - 2) = 3 \cdot 2^{2/3} \simeq 4.76$
- $f(4) = 4^{2/3}(5 - 4) = 4^{2/3} \simeq 2.52$

So the maximum value of f is $y = 6$ at $x = -1$; and the minimum value of f is $y = 0$ at $x = 0$.

7. [8 marks] Find the point (a, a^3) on the graph of $f(x) = x^3$ such that the tangent to the graph of $y = f(x)$ at $(x, y) = (a, a^3)$ passes through the point $(x, y) = (1, 5)$.

Solution: This is Question 55 from Section 3.2 of the textbook. $f'(x) = 3x^2$, so the equation of the tangent line to the graph of $y = f(x)$ at the point (a, a^3) is

$$\frac{y - a^3}{x - a} = f'(a) \Leftrightarrow \frac{y - a^3}{x - a} = 3a^2.$$

For this tangent line to go through the point $(x, y) = (1, 5)$, substitute $x = 1, y = 5$ and solve for a :

$$\begin{aligned} \frac{5 - a^3}{1 - a} = 3a^2 \quad &\Leftrightarrow \quad 5 - a^3 = (1 - a)(3a^2) \\ &\Leftrightarrow \quad 5 - a^3 = 3a^2 - 3a^3 \\ &\Leftrightarrow \quad 2a^3 - 3a^2 + 5 = 0 \\ &\Leftrightarrow \quad (a + 1)(2a^2 - 5a + 5) = 0 \\ &\Rightarrow \quad a = -1 \text{ is the only real solution.} \end{aligned}$$

So the required point on the graph is $(x, y) = (-1, -1)$.