MAT186H1F - Calculus I - Fall 2019

Solutions to Term Test 2 - November 26, 2019

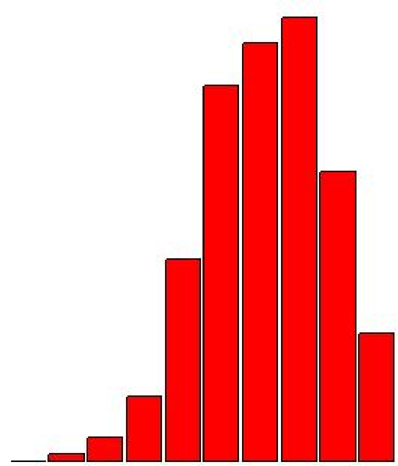
Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

<u>Comments:</u>

- The average on this test was almost identical to the average on the (adjusted) first term test. That means that for anybody who has been excused from either term test during the course, we will simply double the test you did write when calculating your term mark.
- The only question with a failing average was Question 8. Why this should be is hard to fathom—the question is a basic optimization problem, albeit with a slight twist: given the maximum area of the rectangle you have to find the radius. More routine would be to maximize the area of the rectangle given the radius . . . .
- In the attached solutions there are alternate solutions given for some questions. Note that in some cases, e.g. Question 7, the alternate solution is actually much shorter.

Breakdown of Results: 841 registered students wrote this test. The marks ranged from 17.5% to 100%, and the average was 52.9/80 or 66.1%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.5%
A	21.4%	80-89%	14.9%
B	22.7%	70-79%	22.7%
C	21.4%	60-69%	21.4%
D	19.3%	50-59%	19.3%
F	15.2%	40-49%	10.3%
		30-39%	3.3%
		20-29%	1.2%
		10-19%	0.4%
		0-9%	0.0%



1. [avg: 7.6/10] Find the following:

   (a) [3 marks] $\lim_{x\to 0} \dfrac{\tan^{-1}(x)}{\tan(2x)}$

   Solution: this limit is in the 0/0 form so you can apply L'Hôpital's Rule directly:

$$\lim_{x\to 0} \frac{\tan^{-1}(x)}{\tan(2x)} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{2\sec^2(2x)} = \frac{1}{2\sec^2(0)} = \frac{1}{2}$$

   (b) [4 marks] $\lim_{x\to 0} (\cos x)^{1/x^2}$

   Solution: this limit is in the 1^∞ form, so take the logarithm of the limit, and then rearrange it so that you can apply L'Hôpital's Rule.

$$\begin{aligned} L = \lim_{x\to 0} (\cos x)^{1/x^2} \Rightarrow \ln L &= \lim_{x\to 0} \ln\left((\cos x)^{1/x^2}\right) \\ &= \lim_{x\to 0} \frac{\ln\cos x}{x^2} \\ \text{(L'Hôpital's Rule)} &= \lim_{x\to 0} \frac{-\tan x}{2x} \\ \text{(L'Hôpital's Rule)} &= \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2} \\ \Rightarrow L &= e^{-1/2} \text{ or } \frac{1}{\sqrt{e}} \end{aligned}$$

   (c) [3 marks] $(f^{-1})'(e)$, if $f(x) = x\, e^x$ for $x \geq 0$.

   Solution: differentiate implicitly. If $y = xe^x$, then for the inverse $x = ye^y$. So

$$1 = \frac{dx}{dx} = \frac{d(ye^y)}{dx} = \frac{dy}{dx}e^y + ye^y\frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = \frac{1}{e^y + ye^y},$$

   and

$$\left.\frac{dy}{dx}\right|_{x=e} = \left.\frac{1}{e^y + ye^y}\right|_{y=1} = \frac{1}{2e}.$$

   OR: use

$$(f^{-1})'(e) = \frac{1}{f'(f^{-1}(e))}. \ (*)$$

   We have

$$f'(x) = 1\, e^x + x\, e^x \text{ and } f(1) = e \Leftrightarrow 1 = f^{-1}(e);$$

   thus

$$(f^{-1})'(e) = \frac{1}{f'(1)} = \frac{1}{2e}.$$

   Aside: the derivation of $(*)$ starts with $(f \circ f^{-1})(x) = x$. Then by the chain rule,

$$1 = (f \circ f^{-1})'(e) = f'(f^{-1}(e))\,(f^{-1})'(e).$$

2. [20 marks; 15.5/20] This question covers two pages and has eight parts. Let $f(x) = (x^3 + 9x^2)^{1/3} - x$, for which you may assume

$$f'(x) = \frac{x+6}{x^{1/3}(x+9)^{2/3}} - 1, \quad f''(x) = \frac{-18}{x^{4/3}(x+9)^{5/3}}.$$

(a) [1 marks] Simplify $f'(-8)$ and show that it is 0.

Solution: $f'(-8) = \dfrac{-8+6}{(-8)^{1/3}(-8+9)^{2/3}} - 1 = \dfrac{(-2)}{(-2)(1)} - 1 = 0$

(b) [3 marks] Given that $x = -8$ is the *only* value of x for which $f'(x) = 0$ determine how many critical points f has. What are they?

Solution: $f'(x)$ is undefined when $x = 0$ or $x = -9$. So there are three critical points, at $x = -9, -8, 0$. The actual points are

$$(-9, 9), \ (-8, 12), \ (0, 0).$$

(c) [4 marks] What is the sign of $f'(x)$ on the given intervals? Circle your choice.

on the interval	$(-\infty, -9)$	$(-9, -8)$	$(-8, 0)$	$(0, \infty)$
$f'(x)$ is	$\boxed{\text{positive}}$ negative	$\boxed{\text{positive}}$ negative	positive $\boxed{\text{negative}}$	$\boxed{\text{positive}}$ negative

(d) [3 marks] What is the sign of $f''(x)$ on the given intervals? Circle your choice.

on the interval	$(-\infty, -9)$	$(-9, 0)$	$(0, \infty)$
$f''(x)$ is	$\boxed{\text{positive}}$ negative	positive $\boxed{\text{negative}}$	positive $\boxed{\text{negative}}$

For this question

$$f(x) = (x^3 + 9x^2)^{1/3} - x,\ f'(x) = \frac{x+6}{x^{1/3}\,(x+9)^{2/3}} - 1,\ f''(x) = \frac{-18}{x^{4/3}\,(x+9)^{5/3}}.$$

(e) [2 marks] How many inflection points does f have? What are they?

Solution: only one, at $(-9, 9)$.

(f) [4 marks] Find $\lim\limits_{x\to\infty} f(x)$. Hint: $f(x) = x(1 + 9/x)^{1/3} - x$.

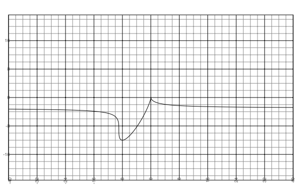
Solution: $f(x) = x\left((1+9/x)^{1/3} - 1\right)$, which is in the $\infty \cdot 0$ form as $x \to \infty$. Thus

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} x\left((1+9/x)^{1/3} - 1\right) \\
&= \lim_{x\to\infty} \frac{\left((1+9/x)^{1/3} - 1\right)}{1/x} \\
\text{(L'H)} \quad &= \lim_{x\to\infty} \frac{\frac{1}{3(1+9/x)^{2/3}}(-9/x^2)}{-1/x^2} \\
&= \lim_{x\to\infty} \frac{3}{(1+9/x)^{2/3}} \\
&= 3
\end{aligned}$$

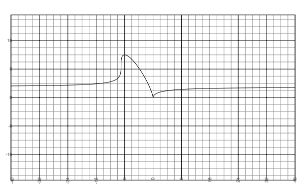
(g) [2 marks] Given that $\lim\limits_{x\to-\infty} f(x)$ is the same as $\lim\limits_{x\to\infty} f(x)$, what are the maximum and minimum values of f?

Solution: $f(0) = 0$ is the absolute minimum of f; $f(-8) = 12$ is the absolute maximum of f.

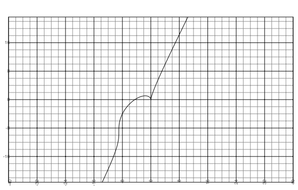
(h) [1 mark] Which of the following is the graph of f? The centre of each picture is the origin.



A



$\boxed{\text{B}}$



C

D

3. [avg: 5.8/10] Suppose that y is a differentiable function of x defined implicitly by the equation

$$(1 + x^3)\, y^2 - 2 = \frac{1}{2} \int_x^{x\,y} \sqrt{1 + 3t^2}\, dt. \quad (*)$$

(a) [3 marks] Find all values of x that satisfy the equation $(*)$ if $y = 1$.

Solution: let $y = 1$, then

$$1 + x^3 - 2 = \frac{1}{2} \int_x^{x} \sqrt{1 + 3t^2}\, dt = 0$$

$$\Rightarrow x^3 = 1 \Rightarrow x = 1.$$

(b) [7 marks] Find the value(s) of $\dfrac{dy}{dx}$ if $y = 1$.

Solution: differentiate implicitly. Use the Fundamental Theorem of Calculus to differentiate the integral on the right side of the equation:

$$3x^2y^2 + (1 + x^3)2y\,\frac{dy}{dx} = \frac{1}{2}\sqrt{1 + 3(xy)^2}\left(y + x\frac{dy}{dx}\right) - \frac{1}{2}\sqrt{1 + 3x^2}.$$

Now substitute $x = 1, y = 1$ and solve for $\dfrac{dy}{dx}$:

$$\begin{aligned}
3 + 4\frac{dy}{dx} &= \frac{1}{2}\sqrt{4}\left(1 + \frac{dy}{dx}\right) - \frac{1}{2}\sqrt{4} \\
\Leftrightarrow 3 + 4\frac{dy}{dx} &= 1 + \frac{dy}{dx} - 1 = \frac{dy}{dx} \\
\Leftrightarrow 3\frac{dy}{dx} &= -3 \\
\Leftrightarrow \frac{dy}{dx} &= -1.
\end{aligned}$$

Aside: theoretically it is possible to integrate $\sqrt{1 + 3t^2}$ with respect to t, but it's *not* easy, and the answer is quite messy:

$$\int \sqrt{1 + 3t^2}\, dt = \frac{t\sqrt{1 + 3t^2}}{2} + \frac{\ln(\sqrt{3}\,t + \sqrt{1 + 3t^2})}{2\sqrt{3}} + C.$$

Then you would have to substitute the limits of integration, $t = x$ and $t = x\,y$, only to have to differentiate the whole mess! So ... using the Fundamental Theorem of Calculus is really the only way to go.

4. [avg: 6.0/10]

4.(a) [5 marks] Let $f(x) = x^2\sqrt{1-x^3}$. Find the average value of f on the interval $[-2, 1]$.

Solution: the average value of f on the interval $[-2, 1]$ is

$$f_{avg} = \frac{1}{(1-(-2))}\int_{-2}^{1} x^2\sqrt{1-x^3}\,dx.$$

Let $u = 1 - x^3$. Then $du = -3x^2\,dx$ and

$$f_{avg} = \frac{1}{3}\frac{1}{3}\int_{9}^{0}\sqrt{u}\,(-du) = \frac{1}{9}\int_{0}^{9}\sqrt{u}\,du = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_0^9 = \frac{2}{27}\,9^{3/2} = 2.$$

4.(b) [5 marks] Find $\displaystyle\int \sin(\sqrt{x})\,dx$.

Solution: this was a tutorial/suggested homework exercise. First let $t = \sqrt{x}$ so that $dt = \dfrac{dx}{2\sqrt{x}}$. Then

$$\int \sin(\sqrt{x})\,dx = \int \sin t\,(2t\,dt) = \int 2t\,\sin t\,dt.$$

Now use integration by parts; let $u = 2t,\ dv = \sin t\,dt$. Then $du = 2\,dt$ and $v = -\cos t$, so:

$$\begin{aligned}
\int \sin(\sqrt{x})\,dx = \int 2t\,\sin t\,dt &= u\,v - \int v\,du \\
&= -2\,t\,\cos t + \int 2\cos t\,dt \\
&= -2\,t\,\cos t + 2\sin t + C \\
\text{(in terms of } x\text{)} &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C
\end{aligned}$$

5. [avg: 6.1/10] When a spherical snowball of initial radius 4 cm starts melting in the sun, the rate at which the radius of the snowball changes is given by

$$\frac{dr}{dt} = -\frac{16}{(t+1)^2},$$

with the radius r measured in cm and the time t measured in hours.

(a) [5 marks] What is the radius of the snowball t hours after it starts melting? How long will it take to melt?

Solution: integrating to find r, (and using substitution $u = 1 + t$, if necessary) you get

$$r = -\int \frac{16}{(t+1)^2}\,dt = \frac{16}{1+t} + C.$$

To find C use the initial condition, $r = 4$ when $t = 0 : 4 = 16 + C \Leftrightarrow C = -12$. Thus

$$r = \frac{16}{1+t} - 12.$$

The snowball has melted when $r = 0$:

$$0 = \frac{16}{1+t} - 12 \Rightarrow 1 + t = \frac{16}{12} \Rightarrow t = \frac{4}{3} - 1 = \frac{1}{3}.$$

That is, it will take one-third of an hour, or 20 minutes, for the snowball to melt.

(b) [5 marks] At what rate is the volume of the snowball decreasing when the radius of the snowball is 1 cm? (The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.)

Solution: from part (a), $r = \dfrac{16}{1+t} - 12$. When $r = 1$ we have

$$1 = \frac{16}{1+t} - 12 \Leftrightarrow 1 + t = \frac{16}{13} \underbrace{\Leftrightarrow t = \frac{3}{13}}_{\text{not needed}}.$$

Differentiating,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So when $r = 1$ and $1 + t = \dfrac{16}{13}$ the volume of the snowball is decreasing at the rate

$$\frac{dV}{dt} = 4\pi(1)^2 \left(-\frac{16}{(16/13)^2}\right) = -\frac{169\,\pi}{4}$$

cubic centimetres per hour.

6. [avg: 8.4/10]

6.(a) [4 marks] Use the method of linear approximation to find an approximate value for $\dfrac{1}{\sqrt{0.97}}$.

(An answer with no supporting calculations will not receive any marks.)

Solution: let $f(x) = x^{-1/2}$, let $a = 1$. Then $f'(x) = -\dfrac{1}{2}x^{-3/2}$ and $f(a) = 1;\ f'(a) = -\dfrac{1}{2}$. Thus

$$f(0.97) \approx f(a) + f'(a)(0.97 - a) = 1 - \frac{1}{2}(0.97 - 1) = 1 - \frac{1}{2}(-0.3) = 1.015.$$

So a linear approximation of $\dfrac{1}{\sqrt{0.97}}$ is 1.015.

Alternate Solution: recall, as seen in WeBWorK Problem Set 5, that $(1+x)^k \approx 1 + k\,x$. You can use this as follows. Take $k = -1/2$ and $x = -0.03$. Thus

$$\frac{1}{\sqrt{0.97}} = (1 - 0.03)^{-1/2} \approx 1 + \left(\frac{-1}{2}\right)(-0.03) = 1.015.$$

6.(b) [6 marks] Use Newton's method to approximate, correct to 5 decimal places, the *largest* solution to the equation $e^x = 6x$.

Solution: the graphs of $y = e^x$ and $y = 6x$ are shown to the right. You can see that there are two solutions to the equation $e^x = 6x$, one near 0 and one near 3. OR, if you let

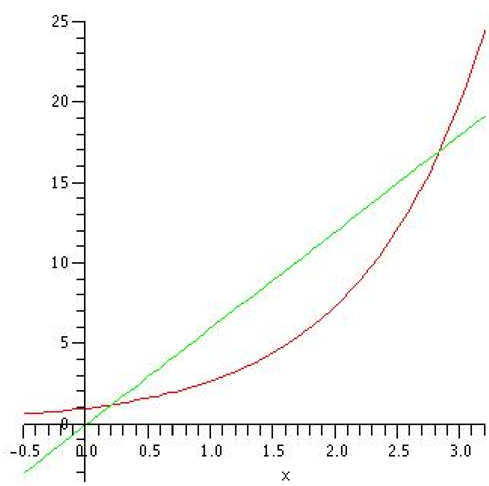
$$f(x) = e^x - 6x$$

you can use your calculator to find

$$f(0) > 0, f(1) < 0, f(2) < 0, f(3) > 0.$$

Thus there are solutions to the equation in the intervals $(0, 1)$ and $(2, 3)$. Either way, use Newton's recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n \geq 0.$$



with $f(x) = e^x - 6x$ and $f'(x) = e^x - 6$. Everything depends on x_0. Pick $x_0 = 3$. Then

$$x_1 = 2.851937705\ldots,\ x_2 = 2.833416292\ldots,\ x_3 = 2.833147947\ldots,\ x_4 = 2.833147892\ldots$$

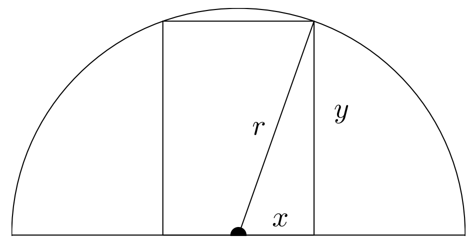
So correct to five decimal places the largest solution to the equation $e^x = 6x$ is

$$x \approx 2.83315.$$

Aside: if your first choice is $x_0 = 0$, say, then you will approximate the other solution, namely

$$x \approx 0.20448.$$

7. [avg: 3.5/10] Design a semicircle, by picking its radius r, so that the largest rectangle that can fit inside it has area 18. One side of the rectangle will lie along the diameter of the semicircle. See figure to the right. Make sure you fully justify that the maximum area achieved for your choice is 18, using relevant course concepts.



Solution: let the base of the rectangle have width $2x$, let the height of the rectangle be y. (See the figure.) Then the area of the rectangle is $A = 2xy$. By the (good old) Pythagorean theorem,

$$x^2 + y^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2}.$$

We now need to find the value of r such that the maximum value of $A = 2x\sqrt{r^2 - x^2}$ on the interval $(0, r)$ is 18. Find the critical point of A :

$$\begin{aligned}
\frac{dA}{dx} &= 2\sqrt{r^2 - x^2} + 2x\left(\frac{-2x}{2\sqrt{r^2 - x^2}}\right) \\
&= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} \\
&= \frac{2r^2 - 4x^2}{\sqrt{r^2 - x^2}}. \\
\text{So } \frac{dA}{dx} = 0 \Rightarrow x &= \frac{r}{\sqrt{2}}, \text{ the only solution in } (0, r)
\end{aligned}$$

You must confirm that this critical point gives a maximum value of A. The easiest way is with the first derivative test:

$$\frac{dA}{dx} > 0 \Leftrightarrow 2r^2 - 4x^2 > 0 \Leftrightarrow x^2 < \frac{r^2}{2} \Leftrightarrow -\frac{r}{\sqrt{2}} < x < \frac{r}{\sqrt{2}};$$

$$\frac{dA}{dx} < 0 \Leftrightarrow 2r^2 - 4x^2 < 0 \Leftrightarrow x^2 > \frac{r^2}{2} \Leftrightarrow x < -\frac{r}{\sqrt{2}} \text{ or } x > \frac{r}{\sqrt{2}}.$$

So on the interval $(0, r)$, A is increasing on $(0, r/\sqrt{2})$ and decreasing on $(r/\sqrt{2}, r)$. Thus the maximum value of A occurs at $x = \dfrac{r}{\sqrt{2}}$ and the maximum value of A is

$$A = \frac{2r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2.$$

Finally, the area of the largest rectangle is to be 18, so take

$$r = \sqrt{18} = 3\sqrt{2}.$$

Aside: you could use the second derivative test to confirm the critical point of A produces a maximum, but then you would have to calculate the second derivative, which is quite messy. See page 10 for this calculation.

Additional Calculations :

For Question 7, $\dfrac{dA}{dx} = \dfrac{2r^2 - 4x^2}{\sqrt{r^2 - x^2}}$, so

$$\begin{aligned}
\frac{d^2A}{dx^2} &= \frac{(-8x)\sqrt{r^2-x^2} - (2r^2-4x^2)(-x/\sqrt{r^2-x^2})}{r^2-x^2} \\
&= \frac{(-8x)(r^2-x^2) - (2r^2-4x^2)(-x)}{(r^2-x^2)^{3/2}} \\
&= \frac{-8xr^2 + 8x^3 + 2xr^2 - 4x^3}{(r^2-x^2)^{3/2}} \\
&= \frac{4x^3 - 6xr^2}{(r^2-x^2)^{3/2}} \\
&= \frac{2x(2x^2-3r^2)}{(r^2-x^2)^{3/2}},
\end{aligned}$$

which is negative when $x^2 = \dfrac{r^2}{2}$, since then

$$2x^2 - 3r^2 = r^2 - 3r^2 = -2r^2 < 0.$$

Or if you completely simplified the second derivative of A at $x = \dfrac{r}{\sqrt{2}}$ you would get

$$\frac{d^2A}{dx^2} = -8 < 0.$$

Either way, by the second derivative test, A has a maximum value at the critical point.

Also in Question 7, it is easier if you square A :

$$A^2 = 4x^2(r^2 - x^2) = 4x^2r^2 - 4x^4.$$

Then you could differentiate implicitly to find the critical point:

$$2A\frac{dA}{dx} = 8r^2x - 16x^3 = 8x(r^2 - 2x^2) = 0 \Rightarrow x = \frac{r}{\sqrt{2}},$$

since $x > 0$.

Alternate Solutions:

For Question 7: let θ be the angle the radius makes with the x-axis. Then

$$A = (2r\,\cos\theta)(r\,\sin\theta) = r^2\sin(2\theta).$$

Then it's obvious that the maximum value of A is at $\theta = \pi/4$, and the maximum value is simply r^2. Finally,

$$r^2 = 18 \Leftrightarrow r = 3\sqrt{2}.$$

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.