# MAT186H1F - Calculus I - Fall 2016

## Solutions to Term Test 2 - November 22, 2016

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

**General Comments:**

1.

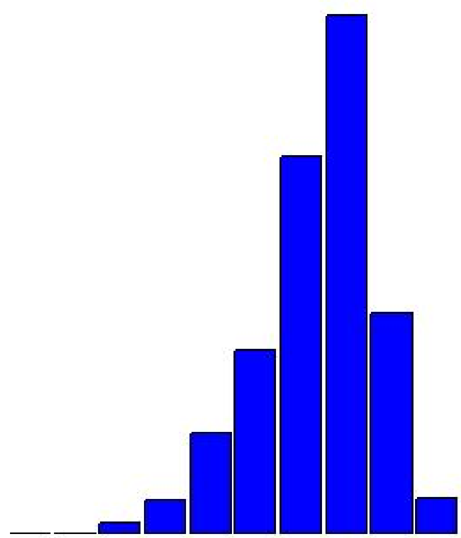
2.

3.

4.

**Breakdown of Results:** 748 students wrote this test. The marks ranged from 25% to 97.5%, and the average was 68.0%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 2.4% |
| A | 17.2% | 80-89% | 14.8% |
| B | 35.0% | 70-79% | 35.0% |
| C | 25.5% | 60-69% | 25.5% |
| D | 12.4% | 50-59% | 12.4% |
| F | 9.8% | 40-49% | 6.8% |
| | | 30-39% | 2.3% |
| | | 20-29% | 0.7% |
| | | 10-19% | 0.0% |
| | | 0-9% | 0.0% |



1. [avg: 6.6/10] Indicate if the following statements are True or False. No justification is required; 1 mark for each correct choice.

(a) If $f$ is continuous on $[a, b]$ then it has an absolute maximum and an absolute minimum on $[a, b]$.

⊕ **True** ○ **False**

(b) A point $c$ is a critical point of $f$ if and only if $f'(c) = 0$. ○ **True** ⊕ **False**

(c) If $f$ is continuous and has a local extremum at $x = c$, then $f$ does not have an inflection point at $x = c$. ○ **True** ⊕ **False**

(d) $f(x) = x^2$ has an absolute minimum on $(-1, 1)$ but no absolute maximum. ⊕ **True** ○ **False**

(e) Suppose $F$ and $G$ are antiderivatives of $f$ on $[0, 4]$. If $F(0) = G(0) + 1$ then $F(4) = G(4) + 5$.

○ **True** ⊕ **False**

(f) Suppose $F$ is an antiderivative of the continuous function $f$ on $[-1, 1]$ and $G(x) = \int_{-1}^{x} f(t)dt$ for $x \geq -1$. If $F(-1) = 7$ then $F(x) = G(x) + 7$. ⊕ **True** ○ **False**

(g) The function $N(t) = \int_0^t e^{-w^2} dw$ is increasing for all $t > 0$. ⊕ **True** ○ **False**

(h) $\lim_{\theta \to \infty} \frac{\theta + \cos\theta}{\theta}$ does not exist. ○ **True** ⊕ **False**

(i) $\lim_{u \to 0} \frac{\tan^{-1} u}{u} = 1$. ⊕ **True** ○ **False**

(j) If $f$ and $g$ are continuous functions, then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}$, if the limits both exist.

⊕ **True** ○ **False**

2. [avg: 7.7/10] Suppose $g(t)$ is continuous everywhere and all the derivatives of $g(t)$ exist except at $t = 4$ and $t = 6$. The following data for $g(t), g'(t)$, and $g''(t)$ on $0 \le t \le 10$ are known:

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $g(t)$ | $-4.1$ | $-0.9$ | 2.2 | 1.4 | 0.4 | 2.6 | 3.1 | 1.3 | $-0.3$ | $-1.2$ | $-3.7$ |
| $g'(t)$ | 3.5 | 3.3 | 0 | $-0.8$ | DNE | 1.6 | DNE | $-1.9$ | 0 | $-0.9$ | $-2.6$ |
| $g''(t)$ | $-0.2$ | $-1.1$ | $-0.2$ | $-0.1$ | DNE | $-1.2$ | DNE | 1.1 | 0 | $-0.3$ | $-0.2$ |

Given that the table contains *all* values $t$, $0 \le t \le 10$, for which $g'(t) = 0$ or $g''(t) = 0$, and *all* values $t$, $0 \le t \le 10$, for which $g'(t)$ or $g''(t)$ does not exist (DNE), find the following for $g$ on $[0, 10]$ :

(a) [3 marks] the values of $t$ for which $g$ is decreasing. (Express your answer in interval notation, set notation, or using inequalities, whichever you prefer.)

**Solution:** need intervals on which $g'(t) < 0$. Including end points is optional.

- In terms of open intervals: $t \in (2, 4) \cup (6, 8) \cup (8, 10)$
- In terms of closed intervals: $t \in [2, 4] \cup [6, 10]$

(b) [2 marks] the absolute maximum and absolute minimum values of $g$.

**Solution:** the extreme values occur at an endpoint, $t = 0$ or $t = 10$, or a critical point, $t = 2, t = 4, t = 6$ or $t = 8$. Compare the values of $g$ :

- $\boxed{\text{the maximum value of } g \text{ is } 3.1}$ (at $t = 6$), and
- $\boxed{\text{the minimum value of } g \text{ is } -4.1}$ (at $t = 0$.)

(c) [3 marks] the values of $t$ for which $g$ is concave down. (Express your answer in interval notation, set notation, or using inequalities, whichever you prefer.)

**Solution:** need intervals on which $g''(t) < 0$. Including end points is optional.

- In terms of open intervals: $t \in (0, 4) \cup (4, 6) \cup (8, 10)$
- In terms of closed intervals: $t \in [0, 6] \cup [8, 10]$

(d)[2 marks] all the inflection points of $g$.

**Solution:** need points at which $g''(t) < 0$ on one side, but $g''(t) > 0$ on the other side. The two inflection points are

- $(6, 3.1)$
- $(8, -0.3)$

3. [avg: 7.1/10] Find and simplify the derivative of the following functions at the point $x = 4$ :

(a) [4 marks] $F(x) = \int_{\pi}^{\sqrt{x}} \sec^{-1} t \, dt$

**Solution:** use the Fundamental Theorem of Calculus, Part 1, and the chain rule:

$$F'(x) = \sec^{-1}\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

So

$$F'(4) = \left(\sec^{-1} 2\right)\left(\frac{1}{4}\right) = \left(\frac{\pi}{3}\right)\left(\frac{1}{4}\right) = \frac{\pi}{12}$$

(b) [6 marks] $G(x) = \frac{(x^2+9)^{3/2}}{(x^3+36)^2}$

**Solution:** use quotient rule or logarithmic differentiation:

$$\ln G(x) = \ln\left(\frac{(x^2+9)^{3/2}}{(x^3+36)^2}\right) = \frac{3}{2}\ln(x^2+9) - 2\ln(x^3+36)$$

$$\Rightarrow \frac{G'(x)}{G(x)} = \frac{3}{2}\left(\frac{2x}{x^2+9}\right) - 2\left(\frac{3x^2}{x^3+36}\right) = \frac{3x}{x^2+9} - \frac{6x^2}{x^3+36}$$

At $x = 4$,

$$G(4) = \frac{25^{3/2}}{100^2} = \frac{1}{80}$$

and

$$G'(4) = G(4)\left(\frac{12}{25} - \frac{96}{100}\right) = \frac{1}{80}\left(-\frac{48}{100}\right) = -\frac{3}{500} = -0.006$$

4. [avg: 8.4/10] Let $e^{2y} + x = y$.

(a) [4 marks] Find the value of $\frac{dy}{dx}$ at the point $(x, y) = (-1, 0)$.

**Solution:** use implicit differentiation, using the chain rule:

$$2\,e^{2y}\frac{dy}{dx} + 1 = \frac{dy}{dx}$$

At $(x, y) = (-1, 0)$,

$$2\frac{dy}{dx} + 1 = \frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = -1$$

(b) [6 marks] Find the value of $\frac{d^2y}{dx^2}$ at the point $(x, y) = (-1, 0)$.

**Solution:** differentiate implicitly again, using the product rule and the chain rule:

$$2\left(2e^{2y}\frac{dy}{dx}\right)\frac{dy}{dx} + 2e^{2y}\frac{d^2y}{dx^2} + 0 = \frac{d^2y}{dx^2}$$

At $(x, y) = (-1, 0)$ from part (a),

$$\frac{dy}{dx} = -1$$

and so

$$4(-1)^2 + 2\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} \Leftrightarrow \frac{d^2y}{dx^2} = -4$$

5. [avg: 7.5/10] Find the following limits.

(a) [4 marks] $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

**Solution:** the limit is in the 0/0 form; use L'Hopital's rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{e^x - 1}{2x} \\
\text{(L'H again)} &= \lim_{x\to 0} \frac{e^x}{2} \\
&= \frac{1}{2}
\end{aligned}$$

(b) [6 marks] $\lim_{\theta\to\pi/4^-} 3\,(\tan\theta)^{\tan(2\theta)}$

**Solution:** apart from the constant factor 3, the limit is in the $1^\infty$ form. Take the natural log of the limit and then use L'Hopital's rule:

$$\begin{aligned}
L &= \lim_{\theta\to\pi/4^-} (\tan\theta)^{\tan(2\theta)} \\
\Rightarrow \ln L &= \lim_{\theta\to\pi/4^-} \ln(\tan\theta)^{\tan(2\theta)} \\
&= \lim_{\theta\to\pi/4^-} \tan(2\theta)\ln(\tan\theta) \\
&= \lim_{\theta\to\pi/4^-} \frac{\ln(\tan\theta)}{\cot(2\theta)}, \text{ in } \frac{0}{0} \text{ form} \\
\text{(L'H)} &= \lim_{\theta\to\pi/4^-} \frac{\frac{\sec^2\theta}{\tan\theta}}{-2\csc^2(2\theta)} \\
&= -\frac{1}{2}\left(\frac{(\sqrt{2})^2}{1}\right) \\
&= -1 \\
\Rightarrow L &= e^{-1}
\end{aligned}$$

So the final answer is

$$\lim_{\theta\to\pi/4^-} 3\,(\tan\theta)^{\tan(2\theta)} = \frac{3}{e}$$

6. [avg: 2.0/10] Suppose that a spherical snowball melts so that its volume decreases at a rate proportional to its surface area. (Recall: for a sphere of radius $r$ its volume is $V = \dfrac{4\pi r^3}{3}$ and its surface area is $S = 4\pi r^2$.) If the volume of the snowball is initially 1000 cm$^3$ when it starts to melt, and its volume is 800 cm$^3$ after 10 seconds, how long will it take to completely melt? (Give your answer to the nearest second.)

**Solution:** there is a constant $k$ such that

$$\frac{dV}{dt} = kS \Leftrightarrow \frac{dV}{dt} = 4k\pi r^2.$$

On the other hand, using the chain rule:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Compare these two equations and conclude that

$$\frac{dr}{dt} = k.$$

Therefore $r = kt + r_0$, where $r_0$ is the radius of the snowball at $t = 0$. At $t = 0$,

$$1000 = \frac{4\pi}{3} r_0^3 \Leftrightarrow r_0 = \left(\frac{750}{\pi}\right)^{1/3}.$$

At $t = 10$,

$$800 = \frac{4\pi}{3} r_{10}^3 \Leftrightarrow r_{10} = \left(\frac{600}{\pi}\right)^{1/3}.$$

Thus

$$r_{10} = r_0 + 10k \Leftrightarrow k = \frac{r_{10} - r_0}{10}$$

Finally the snowball is totally melted when $r = 0$

$$\Leftrightarrow 0 = r_0 + kt \Leftrightarrow t = -\frac{r_0}{k} = \frac{10\, r_0}{r_0 - r_{10}} = \frac{10}{1 - r_{10}/r_0} = \frac{10}{1 - (0.8)^{1/3}} \approx 139.5$$

So it will take 139 (or 140, accept either) seconds for the snowball to melt. You could also say it will take about 2 min and 20 sec.

7. [avg: 8.5/10] The parts of this question are unrelated.

(a) [4 marks] Find the value of $\displaystyle\int_1^2 \left(\frac{x^5-1}{x}\right)\, dx$.

**Solution:** use Fundamental Theorem of Calculus, Part 2:

$$\int_1^2 \left(\frac{x^5-1}{x}\right) dx = \int_1^2 \left(x^4 - \frac{1}{x}\right) dx = \left[\frac{x^5}{5} - \ln x\right]_1^2 = \frac{32}{5} - \ln 2 - \frac{1}{5} = \frac{31}{5} - \ln 2$$

(b) [6 marks] Approximate the solution to the equation $x^3 - x - 2 = 0$ by using Newton's method, starting with $x_0 = 2$, and calculating until the first four decimals of your approximations stop changing.

**Solution:** let $f(x) = x^3 - x - 2$; then $f'(x) = 3x^2 - 1$ and Newtons's recursive formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^3 - x_n - 2}{3x_n^2 - 1} = \frac{2x_n^3 + 2}{3x_n^2 - 1}, \text{ for } n \geq 0.$$

Now use your calculator and calculate:

$$x_0 = 2 \Rightarrow x_1 = 1.636363636\ldots$$

$$x_1 = 1.636363636\cdots \Rightarrow x_2 = 1.530392052\ldots$$

$$x_2 = 1.530392052\cdots \Rightarrow x_3 = 1.521441465\ldots$$

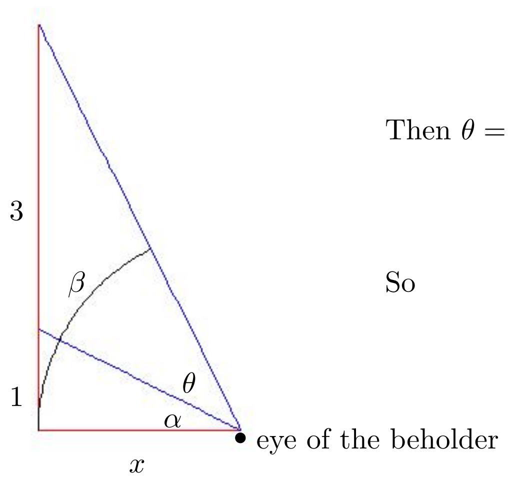
$$x_3 = 1.521441465\cdots \Rightarrow x_4 = 1.521379710\ldots$$

$$x_4 = 1.521379710\cdots \Rightarrow x_5 = 1.521379707\ldots$$

and we can stop since the first four decimal places, actually the first seven, have stopped changing.

8. [avg: 6.6/10] The lower edge of a painting, 3 m in height, is 1 m above an observer's eye level. How far from the wall (on which the painting hangs) should the observer stand to maximize his or her viewing angle?

**Solution:** let the distance from the observer to the wall be $x$, let the angle from eye level to the bottom of the frame be $\alpha$, let the angle from eye level to the top of the frame be $\beta$, let the angle subtended at the observer's eye by the painting be $\theta$.




Then $\theta = \beta - \alpha$ and

$$\tan\alpha = \frac{1}{x}, \quad \tan\beta = \frac{4}{x}.$$

So

$$\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right).$$

The problem is to maximize $\theta$ for $x > 0$. Find the critical point(s):

$$\frac{d\theta}{dx} = \frac{1}{1+(4/x)^2}\left(-\frac{4}{x^2}\right) - \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = -\frac{4}{x^2+16} + \frac{1}{x^2+1}$$

$$\frac{d\theta}{dx} = 0 \Rightarrow \frac{4}{x^2+16} = \frac{1}{x^2+1} = 0 \Rightarrow 4x^2+4 = x^2+16 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

since we are assuming $x > 0$. Confirm a maximum value occurs at $x = 2$:

$$\frac{d^2\theta}{dx^2} = \frac{8x}{(x^2+16)^2} - \frac{2x}{(x^2+1)^2}$$

and

$$\left.\frac{d^2\theta}{dx^2}\right|_{x=2} = -\frac{3}{25} < 0.$$

Conclusion: the observer should stand 2 m from the wall.

**Alternate Solution:** use the cosine law.

$$9 = 16 + x^2 + 1 + x^2 - 2\sqrt{16+x^2}\sqrt{1+x^2}\cos\theta \Rightarrow \cos\theta = \frac{4+x^2}{\sqrt{16+17x^2+x^4}}$$

After much calculation:

$$-\sin\theta\,\frac{d\theta}{dx} = \frac{9x(x^2-4)}{(16+17x^2+x^4)^{3/2}} \text{ and } \frac{d\theta}{dx} = 0 \Rightarrow x = 2,$$

as before.

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