University of Toronto
SOLUTIONS to MAT 186H1F TERM TEST 2
of **Thursday, November 11, 2007**
Duration: 60 minutes
TOTAL MARKS: 50

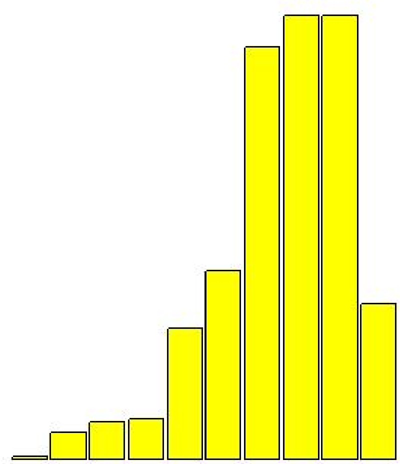
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- All the questions on this test, with one exception, are considered to be very straightforward computations.
- The only exception is Question 5, for which part (a) is simply a statement of a theorem, but for which part (b) requires a little thought to set up.
- The implicit derivatives of Question 4 are easier to evaluate if you substitute the known information as soon as possible, so avoiding having to find general formulas for
$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}.$$
- Some alternate solutions are included.

Breakdown of Results: 548 students wrote this test. The marks ranged from 0% to 98%, and the average was 68.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.2%
A	31.8%	80-89%	23.5%
B	23.5%	70-79%	23.5%
C	21.9%	60-69%	21.9%
D	10.0%	50-59%	10.0%
F	12.8%	40-49%	6.9%
		30-39%	2.2%
		20-29%	2.0%
		10-19%	1.5%
		0-9%	0.2 %



1. [16 marks] This question has five parts and covers two pages. Let $f(x) = \dfrac{2x}{x^2+1}$; for which

$$f'(x) = \frac{2-2x^2}{(x^2+1)^2} \text{ and } f''(x) = \frac{4x^3-12x}{(x^2+1)^3}.$$

(a) [3 marks] Find the open intervals on which f is increasing and those on which it is decreasing.

Solution:

$$\begin{aligned} f'(x) &= \frac{2-2x^2}{(x^2+1)^2} \\ &= 2\frac{(1-x)(1+x)}{(x^2+1)^2} \end{aligned}$$

$$\begin{aligned} f \text{ is increasing if } f'(x) > 0 &\Leftrightarrow (1-x)(1+x) > 0 \\ &\Leftrightarrow -1 < x < 1 \end{aligned}$$

$$\begin{aligned} f \text{ is decreasing if } f'(x) < 0 &\Leftrightarrow (1-x)(1+x) < 0 \\ &\Leftrightarrow x < -1 \text{ or } x > 1 \end{aligned}$$

(b) [4 marks] Find all the critical points of f and determine if they are maximum or minimum points.

Solution: $f'(x) = 0 \Leftrightarrow (1-x)(1+x) = 0 \Leftrightarrow x = \pm 1$.

Using the first derivative test:

1. Since f is decreasing if $x < -1$ and f is increasing if $x > -1$, the point $(-1, f(-1)) = (-1, -1)$ is a minimum point.
2. Since f is increasing if $x < 1$ and f is decreasing if $x > 1$, the point $(1, f(1)) = (1, 1)$ is a maximum point.

Using the second derivative test:

1. Since $f''(-1) = 1 > 0$, the point $(-1, f(-1)) = (-1, -1)$ is a minimum point.
2. Since $f''(1) = -1 < 0$, the point the point $(1, f(1)) = (1, 1)$ is a maximum point.

(c) [4 marks] Find the open intervals on which f is concave up, and those on which it is concave down.

Solution:

$$\begin{aligned} f''(x) &= \frac{4x^3 - 12x}{(x^2+1)^3} \\ &= \frac{4x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3} \end{aligned}$$

$$\begin{aligned} f \text{ is concave up if } f''(x) > 0 &\Leftrightarrow x(x-\sqrt{3})(x+\sqrt{3}) > 0 \\ &\Leftrightarrow -\sqrt{3} < x < 0 \text{ or } x > \sqrt{3} \end{aligned}$$

$$\begin{aligned} f \text{ is concave down if } f''(x) < 0 &\Leftrightarrow x(x-\sqrt{3})(x+\sqrt{3}) < 0 \\ &\Leftrightarrow x < -\sqrt{3} \text{ or } 0 < x < \sqrt{3} \end{aligned}$$

(d) [3 marks] Find all the inflection points of f, if any.

Solution:

$$\begin{aligned} f''(x) = 0 &\Leftrightarrow x(x-\sqrt{3})(x+\sqrt{3}) = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = -\sqrt{3} \text{ or } x = \sqrt{3} \end{aligned}$$

From part (c), there are inflection points at

1. $(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, -\frac{\sqrt{3}}{2})$
2. $(0, f(0)) = (0, 0)$
3. $(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, \frac{\sqrt{3}}{2})$

(e) [2 marks] Find all the horizontal or vertical asymptotes to the graph of f, if any.

Solution: There are no vertical asymptotes since f is continuous for all x.

There is one horizontal asymptote, $y = 0$, since

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

2. [7 marks] Find the following:

(a) [2 marks] $\dfrac{d}{dx}\tan(\cos x)$

Solution: Use the chain rule.

$$\frac{d}{dx}\tan(\cos x) = \sec^2(\cos x)(-\sin x)$$

(b) [5 marks] $\dfrac{d}{dx}(\sec x)^x$. (Assume $\sec x > 0$.)

Solution: Let $y = (\sec x)^x$ and use logarithmic differentiation.

$$\begin{aligned} \ln y = x \ln \sec x \quad &\Rightarrow \quad \frac{y'}{y} = \ln \sec x + x\frac{\sec x \tan x}{\sec x} \\ &\Rightarrow \quad y' = y(\ln \sec x + x \tan x) \\ &\Rightarrow \quad y' = (\sec x)^x (\ln \sec x + x \tan x) \end{aligned}$$

3. [7 marks] Find the following limits.

(a) [3 marks] $\lim_{x \to 0} \dfrac{e^{2x} - 1}{\ln(x+1)}$

Solution: Limit is in the $\dfrac{0}{0}$ form. Use L'Hopital's rule.

$$\begin{aligned}\lim_{x \to 0} \frac{e^{2x} - 1}{\ln(x+1)} &= \lim_{x \to 0} \frac{2e^{2x}}{\frac{1}{x+1}} \\ &= \frac{2}{1}\end{aligned}$$

(b) [4 marks] $\lim_{x \to 0^+} (x + e^{-2x})^{3/x}$

Solution: Limit is in the 1^∞ form. Let the limit be L.

$$\begin{aligned}\ln L &= \lim_{x \to 0^+} \frac{3}{x} \ln\left(x + e^{-2x}\right) \\ &= 3 \lim_{x \to 0^+} \frac{\ln\left(x + e^{-2x}\right)}{x}, \quad \text{which is in } \frac{0}{0} \text{ form} \\ &= 3 \lim_{x \to 0^+} \frac{1 - 2e^{-2x}}{x + e^{-2x}}, \quad \text{by L'Hopital's rule} \\ &= 3\,(1-2) \\ &= -3 \\ \Rightarrow L &= e^{-3}\end{aligned}$$

4. [7 marks] Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (1, 3)$ if

$$xy = 3e^{3x-y}.$$

Solution: Differentiate implicitly.

$$xy' + y = 3e^{3x-y}\,(3 - y')$$

Substiute $x = 1, y = 3:$

$$y' + 3 = 3e^0(3 - y') \Leftrightarrow y' + 3 = 9 - 3y' \Leftrightarrow y' = \frac{3}{2}$$

To find y'', differentiate implicitly once more:

$$y' + xy'' + y' = 3e^{3x-y}\,(3 - y')^2 + 3e^{3x-y}\,(-y'')$$

Now substitute $x = 1, y = 3, y' = \frac{3}{2}:$

$$\begin{aligned} & \frac{3}{2} + y'' + \frac{3}{2} = 3e^0\left(3 - \frac{3}{2}\right)^2 + 3e^0\,(-y'') \\ \Leftrightarrow\quad & 3 + y'' = \frac{27}{4} - 3y'' \\ \Leftrightarrow\quad & y'' = \frac{15}{16} \end{aligned}$$

Alternate Solution: Use logarithmic differentiation.

$$\ln x + \ln y = \ln 3 + 3x - y \Rightarrow \frac{1}{x} + \frac{y'}{y} = 3 - y'$$

Substiute $x = 1, y = 3:$

$$1 + \frac{y'}{3} = 3 - y' \Leftrightarrow y' = \frac{3}{2}$$

To find y'', differentiate implicitly once more:

$$-\frac{1}{x^2} + \frac{yy'' - y'y'}{y^2} = -y''$$

Now substitute $x = 1, y = 3, y' = \frac{3}{2}:$

$$-1 + \frac{3y'' - \frac{9}{4}}{9} = -y'' \Leftrightarrow y'' = \frac{15}{16}$$

5. [6 marks; 3 marks for each part.]

(a) State the Mean Value Theorem.

Solution:

Hypotheses: f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Conclusion: there is a number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) Use the Mean Value Theorem to prove that if $f(x) = (x^2 - x + 9)\cos x + 5x$, then there is a number c such that $f'(c) = 5$.

Solution: f is continuous and differentiable for all values of x, so the Mean Value Theorem applies to f on every possible interval $[a, b]$. Pick the interval $[a, b] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then for some number c in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} \quad &\Leftrightarrow \quad f'(c) = \frac{5\left(\frac{\pi}{2}\right) - 5\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}, \text{ since } \cos\left(\pm\frac{\pi}{2}\right) = 0 \\ &\Leftrightarrow \quad f'(c) = 5 \end{aligned}$$

6. [7 marks]

(a) [3 marks] Find an approximation to $9^{\frac{1}{3}}$ by using the linear approximation of $f(x) = x^{\frac{1}{3}}$ at $a = 8$. (Express your answer to five decimal places.)

Solution: $f'(x) = \dfrac{1}{3}\dfrac{1}{x^{2/3}}$, so the equation of the tangent line to f at $a = 8$ is

$$\begin{aligned} \frac{y - f(8)}{x - 8} = f'(8) \quad &\Leftrightarrow \quad \frac{y-2}{x-8} = \frac{1}{3}\frac{1}{8^{2/3}} \\ &\Leftrightarrow \quad y = 2 + \frac{1}{12}(x - 8). \\ \text{So} \quad 9^{1/3} = f(9) \quad &\simeq \quad 2 + \frac{1}{12}(9 - 8) \\ &= \quad \frac{25}{12} \simeq 2.08333 \end{aligned}$$

(b) [4 marks] Find an approximation to $9^{\frac{1}{3}}$ by applying Newton's method to the equation

$$x^3 - 9 = 0;$$

start with $x_0 = 2$ and compute x_1 and x_2. (Express your answers to five decimal places.)

Solution: $f(x) = x^3 - 9; f'(x) = 3x^2$. So the recursive formula for Newton's method is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 9}{3x_n^2} \\ &= \frac{2x_n^3 + 9}{3x_n^2} \end{aligned}$$

$$x_0 = 2 \;\Rightarrow\; x_1 = \frac{25}{12} \simeq 2.08333$$

and

$$x_1 = \frac{25}{12} \;\Rightarrow\; x_2 = \frac{23401}{11250} \simeq 2.08009$$