University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, December 2019

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT186H1F - Calculus I

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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

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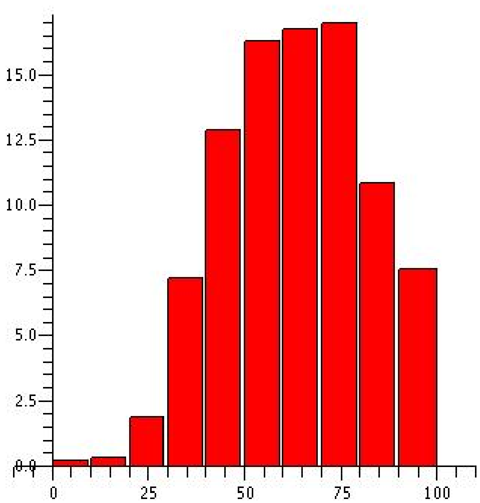
Breakdown of Results: 828 registered students wrote this test. The marks ranged from 2.5% to 100% and the average was 50.58/80 or 63.2%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	9.2%
A	21.0%	80-89%	11.8%
B	18.5%	70-79%	18.5%
C	18.2%	60-69%	18.2%
D	17.8%	50-59%	17.8%
F	24.5%	40-49%	14.0%
		30-39%	7.9%
		20-29%	2.0%
		10-19%	0.4%
		0-9%	0.2%



1. [10 marks; 2 marks for each part. Avg: 7.8/10] Let $f(x) = \sqrt{25 - x^2}$. Write down an integral that gives the value of each of the following quantities. (Do NOT evaluate the integrals.)

(a) The area of the region bounded by the curves with equations $y = f(x),\ y = x,\ x = 0$ and $x = 3$.

Solution:

$$\int_0^3 (f(x) - x)\, dx \quad \text{OR} \quad \int_0^3 (\sqrt{25 - x^2} - x)\, dx$$

(b) The volume of the solid of revolution obtained by rotating around the y-axis the region bounded by the curves with equations $y = f(x),\ y = 0,\ x = 0$ and $x = 3$.

Solution:

$$\underbrace{\int_0^3 2\pi x\, f(x)\, dx}_{\text{method of shells}} \quad \text{OR} \quad \int_0^3 2\pi x \sqrt{25 - x^2}\, dx.$$

Note:

$$\underbrace{\int_4^5 \pi(25 - y^2)\, dy}_{\text{method of discs}} + 36\pi$$

gives correct volume but is not solely an integral.

(c) The length of the curve with equation $y = f(x)$ for $0 \le x \le 3$.

Solution:

$$\int_0^3 \sqrt{1 + (f'(x))^2}\, dx \quad \text{OR} \quad \int_0^3 \frac{5}{\sqrt{25 - x^2}}\, dx.$$

(d) The surface area of the solid of revolution obtained by rotating around the x-axis the curve with equation $y = f(x)$ for $0 \le x \le 3$.

Solution:

$$\int_0^3 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx \quad \text{OR} \quad \int_0^3 10\pi\, dx$$

(e) The surface area of the solid of revolution obtained by rotating around the y-axis the curve with equation $y = f(x)$ for $0 \le x \le 3$.

Solution:

$$\int_0^3 2\pi x \sqrt{1 + (f'(x))^2}\, dx \text{ OR } \int_4^5 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy,$$

where $g(y) = f^{-1}(y) = \sqrt{25 - y^2}$. **Note:** as in part (c), $\sqrt{1 + (f'(x))^2} = \dfrac{5}{\sqrt{25 - x^2}}$.

2. [avg: 9.4/10] Let $v = 2\sqrt{t} - 4$ be the velocity of a particle at time t, for $0 \le t \le 16$. Find:

(a) [4 marks] the average velocity of the particle.

Solution: use the formula for average.

$$\begin{aligned} v_{avg} &= \frac{1}{16-0}\int_0^{16} v\,dt \\ &= \frac{1}{16}\int_0^{16} (2\sqrt{t}-4)\,dt \\ &= \frac{1}{16}\left[\frac{4t^{3/2}}{3} - 4t\right]_0^{16} \\ &= \frac{16}{3} - 4 = \frac{4}{3} \end{aligned}$$

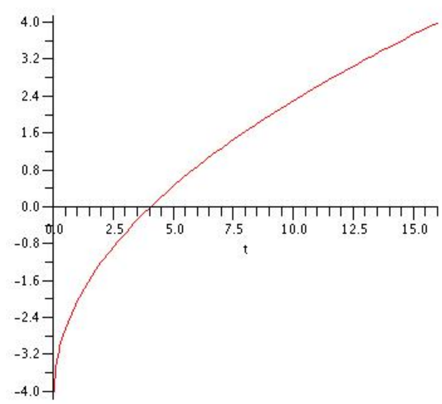
(b) [6 marks] the average speed of the particle.

Solution: since v changes signs on the interval $[0, 16]$ you have to calculate the average speed in two steps. We have

$$v = 0 \Rightarrow 2\sqrt{t} = 4 \Rightarrow t = 4,$$

and so the average speed $s = |v|$ is given by

$$s_{avg} = \frac{1}{16}\int_0^{16} |v|\,dt = \frac{1}{16}\left(\int_0^4 -v\,dt + \int_4^{16} v\,dt\right).$$



Calculations:

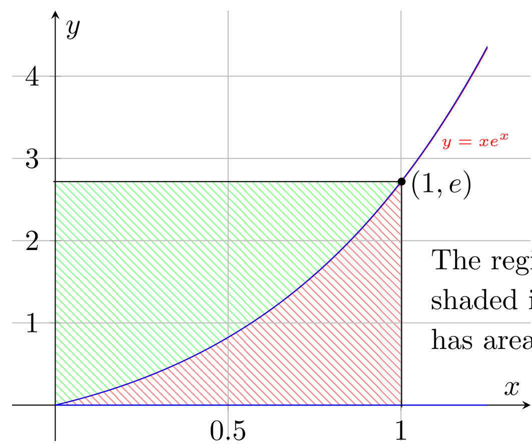
$$\begin{aligned} s_{avg} &= \frac{1}{16}\int_0^4 (4-2\sqrt{t})\,dt + \frac{1}{16}\int_4^{16} (2\sqrt{t}-4)\,dt \\ &= \frac{1}{16}\left[4t - \frac{4t^{3/2}}{3}\right]_0^4 + \frac{1}{16}\left[\frac{4t^{3/2}}{3} - 4t\right]_4^{16} \\ &= \frac{1}{16}\left(16 - \frac{32}{3}\right) + \frac{1}{16}\left(\frac{256}{3} - 64 - \frac{32}{3} + 16\right) \\ &= 1 - \frac{2}{3} + \frac{14}{3} - 3 = \frac{1}{3} + \frac{5}{3} = 2 \end{aligned}$$

3. [avg: 7.2/10] Let $f(x) = x\,e^x$, for $x \geq 0$.

(a) [2 marks] The graph of $y = f(x)$ is shown below. Clearly indicate the regions that have area corresponding to each of $A_1 = \int_0^1 f(x)\,dx$ and $A_2 = \int_0^e f^{-1}(y)\,dy$.

Solution:

The region shaded in green has area A_2



The region shaded in red has area A_1

(b) [4 marks] Find the value of A_1.

Solution: use parts with $u = x$ and $dv = e^x\,dx$. Then $du = dx,\ v = e^x$ and

$$A_1 = \int_0^1 xe^x\,dx = [xe^x]_0^1 - \int_0^1 e^x\,dx = [xe^x]_0^1 - [e^x]_0^1 = e - 0 - e + 1 = 1$$

(c) [4 marks] Find the value of A_2.

Short Way: subtract A_1 from the area of the rectangle bounded by $0 \leq x \leq 1$ and $0 \leq y \leq e$:

$$A_2 = e - A_1 = e - 1.$$

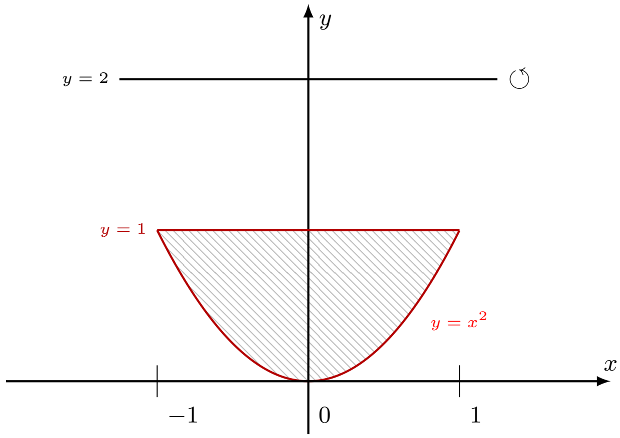
Long Way: let $y = f(x) = xe^x$. Then $f^{-1}(y) = f^{-1}(f(x)) = x$ and $dy = (x+1)e^x\,dx$ and

$$\begin{aligned} A_2 = \int_0^e f^{-1}(y)\,dy &= \int_{f^{-1}(0)}^{f^{-1}(e)} x\,(x+1)e^x\,dx \\ &= \int_0^1 (x^2+x)e^x\,dx \\ \text{(use parts twice; see page 12)} &= \left[(x^2-x+1)e^x\right]_0^1 \\ &= e - 1 \end{aligned}$$

4. [avg: 5.9/10] Find the volume of the solid generated by rotating around the line $y = 2$ the region bounded by the curves with equations $y = x^2$ and $y = 1$, using:

(a) [5 marks] the method of cylindrical shells.

Solution: the region is in the figure below.



Using the method of shells, with respect to y :

$$\begin{aligned} V &= \int_0^1 2\pi(2-y)(\sqrt{y} - (-\sqrt{y}))\,dy \\ &= 4\pi \int_0^1 (2-y)\sqrt{y}\,dy \\ &= 4\pi \int_0^1 \left(2y^{\frac{1}{2}} - y^{\frac{3}{2}}\right)\,dy \\ &= 4\pi \left[\frac{4}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right]_0^1 \\ &= \frac{56\pi}{15} \end{aligned}$$

(b) [5 marks] the method of discs and washers.

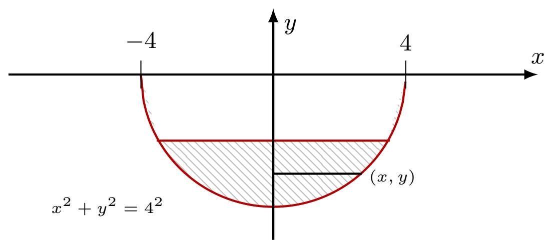
Solution: using the method of discs and integrating with respect to x :

$$\begin{aligned} V &= \int_{-1}^1 \left(\pi(2-x^2)^2 - \pi(1)^2\right)\,dx \\ &= \pi \int_{-1}^1 \left(4 - 4x^2 + x^4 - 1\right)\,dx \\ &= 2\pi \int_0^1 \left(3 - 4x^2 + x^4\right)\,dx \\ &= 2\pi \left[3x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_0^1 \\ &= \frac{56\pi}{15} \end{aligned}$$

5. [avg: 3.9/10] A hemispherical tank of radius 4 m contains water in the bottom 2 m of the tank.

(a) [6 marks] How much work does it take to empty the tank by pumping all the water up to the top of the tank and out? (Assume the density of water is ρ and that the acceleration due to gravity is g; leave your answer in terms of ρ and g.)

Solution: if you set $y = 0$ at the top of the tank, then the equation of the circular side view is $x^2 + y^2 = 4^2$, the bottom of the water is at $a = -4$, and the top of the water is at $b = -2$. The cross-sectional area of the tank at height y is



$$A(y) = \pi x^2 = \pi(4^2 - y^2).$$

Then the work done in pumping all the water from in the tank up to the top $y = 0$ is given by

$$\begin{aligned} W &= \int_a^b \rho\, g\, A(y)(0 - y)\, dy = \int_{-4}^{-2} \rho\, g\, \pi(4^2 - y^2)(0 - y)\, dy \\ &= \rho\, \pi\, g \int_{-4}^{-2} (y^3 - 16y)\, dy = \rho\, \pi\, g \left[\frac{y^4}{4} - 8y^2\right]_{-4}^{-2} = 36\, \rho\, \pi\, g \text{ (Joules)} \end{aligned}$$

(b) [4 marks] Suppose the water is being pumped out at a rate of 1 cubic meter per minute. How fast is the depth of the water decreasing when the depth of the water in the tank is 1 m?

Solution: using the same set up as in part (a), the volume of the water in the tank at depth h is given by

$$V = \int_{-4}^{h-4} A(y)\, dy = \pi \int_{-4}^{h-4} (16 - y^2)\, dy.$$

Then, using the Chain Rule and the Fundamental Theorem of Calculus,

$$\frac{dV}{dt} = \pi(16 - (h-4)^2)\, \frac{dh}{dt}.$$

When $\dfrac{dV}{dt} = -1,\ h = 1$, we have

$$-1 = \pi(16 - 3^2)\, \frac{dh}{dt} \Leftrightarrow \frac{dh}{dt} = -\frac{1}{7\,\pi}.$$

So the depth of the water is decreasing at a rate of $\dfrac{1}{7\,\pi}$ m/min. (Approximately: 4.54 cm/min)

Alternate Solutions: on pages 10 and 11 there are alternate solutions to both parts of this problem.

6. [avg: 5.5/10] Let $g(x) = x^3 + \sin x$.

(a) [4 marks] Prove that g is a one-to-one function.

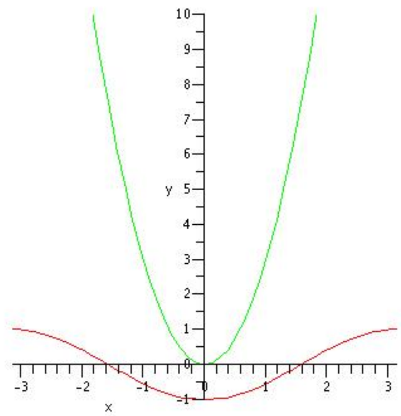
Solution: we shall show g is an increasing function for all x, from which it follows g must be one-to-one. We have

$$g'(x) = 3x^2 + \cos x$$

and

$$g'(x) > 0 \Leftrightarrow 3x^2 > -\cos x.$$

In the figure to the right you can see that the parabola, $y = 3x^2$, is always above the trig function, $y = -\cos x$.



(b) [2 marks] Explain why the equation $g(x) = 4$ has exactly one solution in the interval $[0, 2]$.

Solution: $g(0) = 0 < 4$ and $g(2) = 8 + \sin 2 \geq 7 > 4$, so by IVT there is *at least one* number c in $(0, 2)$ such that

$$g(c) = 4.$$

But by part (a) g is one-to-one, which means there is *at most one* number c such that $g(c) = 4$. Therefore there is exactly one number c in $[0, 2]$ such that $g(c) = 4$.

(c) [4 marks] Approximate the solution to the equation $x^3 + \sin x = 4$ correct to 4 decimal places using Newton's method.

Solution: let $f(x) = x^3 + \sin x - 4$. Then $f'(x) = 3x^2 + \cos x$. Newton's recursive formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n \geq 0.$$

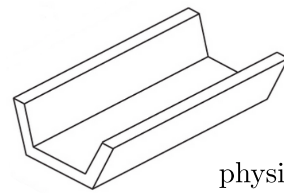
Pick $x_0 = 1$ (although it doesn't really matter what your initial choice is) and calculate[1]:

$$x_1 = 1.609702\ldots,\ x_2 = 1.458405\ldots,\ x_3 = 1.443676\ldots,\ x_4 = 1.443544\ldots,\ x_5 = 1.443544\ldots$$

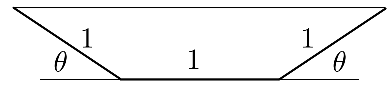
So, correct to 4 decimal places, the solution to the equation $x^3 + \sin x = 4$ is $x = 1.4435$

[1] Of course your calculator must be in radian mode.

7. [avg: 5.8/10] A drainage channel is to be constructed so that its cross section is a trapezoid with equally sloping sides. (See figures to the right.) If the sides and bottom of a cross section all have a length of 1 m, how should the angle between the sides and the bottom be chosen to maximize the cross-sectional area of the channel?

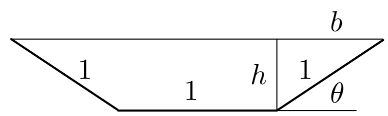


physical model



cross section

Solution: let the base and height of each triangle at the end of the trapezoidal cross section be b and h, respectively. See figure to the right.



cross section

Then the triangle at each end of the trapezoidal cross section has area

$$T = \frac{bh}{2},$$

with $b = \cos\theta$ and $h = \sin\theta$. Thus the total area of the trapezoidal cross section is

$$A = 2T + (1)(h) = \cos\theta\,\sin\theta + \sin\theta = \frac{\sin(2\theta)}{2} + \sin\theta.$$

$\boxed{\text{The problem is to maximize the value of } A \text{ for } 0 < \theta < \frac{\pi}{2}.}$ Calculating derivatives we find

$$\frac{dA}{d\theta} = \cos(2\theta) + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = (2\cos\theta - 1)(\cos\theta + 1); \quad \frac{d^2A}{d\theta^2} = -2\sin(2\theta) - \sin\theta.$$

Critical Points:

$$\frac{dA}{d\theta} = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1.$$

The only critical point in the interval $(0, \pi/2)$ is $\theta = \pi/3$, or $60°$. At this point, $\frac{d^2A}{d\theta^2} = -\frac{3\sqrt{3}}{2} < 0$.

Conclusion:

- To maximize the cross-sectional area of the drainage channel the angle between the sides and the bottom should be $60°$.

8. [avg: 5.0/10] Let $\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$; let $\operatorname{Si}(x) = \displaystyle\int_0^x \operatorname{sinc}(t)\,dt$.

(a) [2 marks] Is $\operatorname{Si}(x)$ a continuous function? Justify your answer.

Solution: yes. Since $\operatorname{sinc}(x)$ is continuous for all x, the definite integral $\operatorname{Si}(x) = \displaystyle\int_0^x \operatorname{sinc}(t)\,dt$ exists for all x, and so by the Fundamental Theorem of Calculus, $\operatorname{Si}'(x) = \operatorname{sinc}(x)$ for all x. This means $\operatorname{Si}(x)$ is differentiable, hence continuous, for all x.

(b) [2 marks] Show that $\operatorname{Si}(x)$ is an odd function.

Solution: let $u = -t$. Then

$$\operatorname{Si}(-x) = \int_0^{-x} \operatorname{sinc}(t)\,dt = \int_0^{x} \operatorname{sinc}(-u)\,(-du) = -\int_0^{x} \operatorname{sinc}(u)\,du = -\operatorname{Si}(x),$$

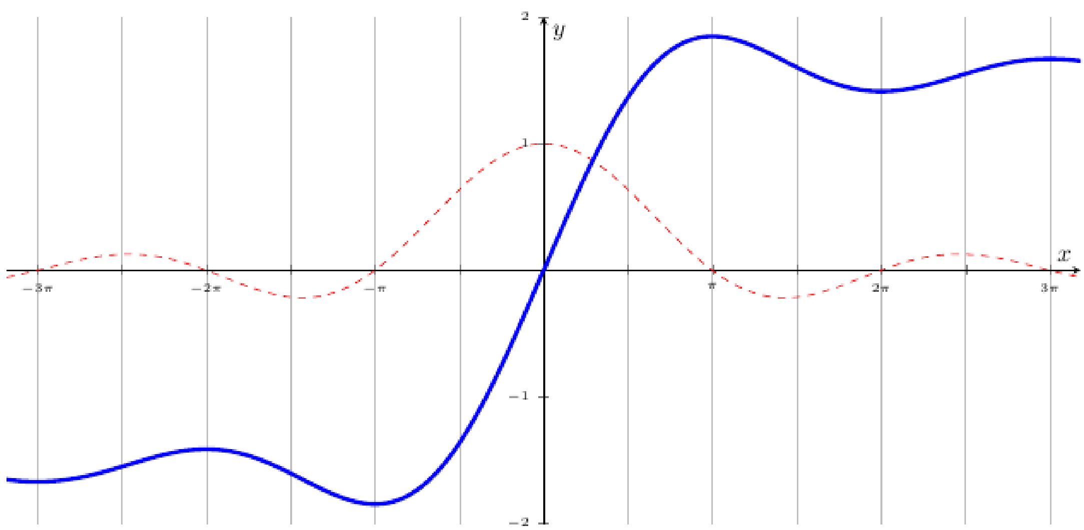
where we have used the fact that $\operatorname{sinc}(x)$ is an even function.

(c) [2 marks] What are the critical points of $\operatorname{Si}(x)$ for $-10 \leq x \leq 10$?

Solution: $\operatorname{Si}'(x) = 0 \Rightarrow \operatorname{sinc}(x) = 0 \Rightarrow x \neq 0$ and $\sin(x) = 0 \Rightarrow x = \pm\pi, \pm 2\pi, \pm 3\pi$.

(d) [4 marks] The graph of $\operatorname{sinc}(x)$ is dotted in below. Sketch in the corresponding graph of $\operatorname{Si}(x)$.

Solution: we have $\operatorname{Si}(0) = 0$. Since $\operatorname{Si}'(x) = \operatorname{sinc}(x)$, we know: Si is increasing when $\operatorname{sinc}(x) > 0$ and Si is decreasing when $\operatorname{sinc}(x) < 0$; Si is concave up when sinc is increasing and Si is concave down when sinc is decreasing. Thus the graph of $y = \operatorname{Si}(x)$ looks like the blue graph below:



Alternate Calculations:

For Question 5, if you put the centre of the circle at $(0, 4)$ then the equation of the circle is

$$x^2 + (y-4)^2 = 4^2 \Leftrightarrow x^2 - 8y + y^2 = 0.$$

In this version the bottom of the water is at $a = 0$, the top of the water is at $b = 2$, and the cross-sectional area at y is

$$A(y) = \pi x^2 = \pi(8y - y^2).$$

So the work done is

$$\begin{aligned} W &= \int_0^2 \rho\, g\, A(y)(4-y)\, dy \\ &= \rho\, g\, \pi \int_0^2 (8y - y^2)(4-y)\, dy \\ &= \rho\, g\, \pi \int_0^2 \left(y^3 - 12y^2 + 32y\right)\, dy \\ &= \rho\, g\, \pi \left[\frac{y^4}{4} - 4y^3 + 16y^2\right]_0^2 \\ &= \rho\, g\, \pi(4 - 32 + 64) \\ &= 36\, \rho\, g\, \pi \end{aligned}$$

For part (b), the depth of the water is h and the volume of this water is

$$V = \int_0^h A(y)\, dy = \pi \int_0^h (8y - y^2)\, dy.$$

By the Chain Rule and the Fundamental Theorem of Calculus,

$$\frac{dV}{dt} = \pi(8h - h^2)\frac{dh}{dt}.$$

Then with $\dfrac{dV}{dt} = -1$ and $h = 1$ we have

$$-1 = \pi(8-1)\frac{dh}{dt} \Leftrightarrow \frac{dh}{dt} = -\frac{1}{7\,\pi},$$

as before.

Question 5: if your tank curves downward, and the tank sits on the x-axis. Then the equation of the side view is

$$x^2 + y^2 = 4^2, \; y \geq 0.$$

For part (a) the work done is now

$$W = \int_0^2 \rho\, g\, A(y)\,(4-y)\,dy = \pi\rho g \int_0^2 (16-y^2)(4-y)\,dy = \pi\rho g \int_0^2 (64 - 16y - 4y^2 + y^3)\,dy = \frac{268}{3}\pi\rho g.$$

For part (b), the volume of water in the tank at depth h is

$$V = \int_0^h A(y)\,dy = \int_0^h \pi(16-y^2)\,dy$$

and

$$\frac{dV}{dt} = \pi(16-h^2)\frac{dh}{dt}.$$

Then when $\dfrac{dV}{dt} = -1, h = 1$, we have

$$-1 = \pi(16-1)\frac{dh}{dt} \Leftrightarrow \frac{dh}{dt} = -\frac{1}{15\pi}.$$

Alternate Calculations:

For Question 3(c):

$$
\begin{aligned}
\int (x^2+x)e^x\,dx &= \int u\,dv, \text{ with } u = x^2+x; dv = e^x\,dx \\
&= uv - \int v\,du \\
&= (x^2+x)e^x - \int (2x+1)e^x\,dx \\
&= (x^2+x)e^x - 2\int xe^x\,dx - e^x \\
&= (x^2+x-1)e^x - 2\int s\,dt, \text{ with } s = x; dt = e^x\,dx \\
&= (x^2+x-1)e^x - 2\left(st - \int t\,ds\right) \\
&= (x^2+x-1)e^x - 2xe^x + 2\int e^x\,dx \\
&= (x^2+x-1)e^x - 2xe^x + 2e^x + C \\
&= (x^2-x+1)e^x + C
\end{aligned}
$$