University of Toronto

Faculty of Applied Science and Engineering

Final Examination, December 2018

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

Solutions to **MAT186H1F - Calculus I**

Examiners: F. Bischoff, D. Burbulla, S. Cohen, D. Fusca, J. Ko, X. Jie
M. Matviichuk, K. Pham

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

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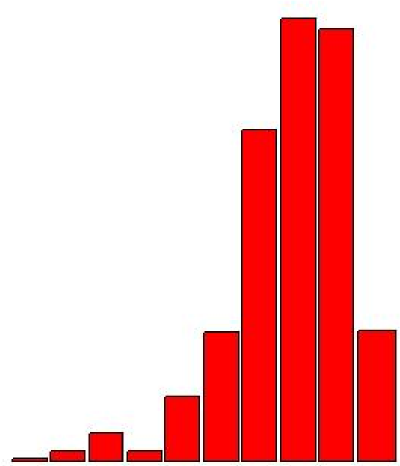
Breakdown of Results: 750 registered students wrote this test. The marks ranged from 7.5% to 100%, and the average was 72.2%. There was 1 perfect paper. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	9.1%
A	36.2%	80-89%	27.1%
B	27.7%	70-79%	27.7%
C	20.8%	60-69%	20.8%
D	8.1%	50-59%	8.1%
F	7.2%	40-49%	4.0%
		30-39%	0.7%
		20-29%	1.7%
		10-19%	0.7%
		0-9%	0.1%



1. [avg: 7.32/10] Find the following:

(a) [3 marks] $\lim_{x\to 1} \dfrac{\ln x}{x^3 - x}$

Solution: the limit is in the 0/0 form so use L'Hopital's rule:

$$\lim_{x\to 1} \frac{\ln x}{x^3 - x} = \lim_{x\to 1} \frac{1/x}{3x^2 - 1} = \frac{1}{2}$$

(b) [3 marks] the equation of the tangent line to the graph of $f(x) = \tan^{-1} x$ at the point $(1, \pi/4)$.

Solution: $f'(x) = \dfrac{1}{1+x^2}$ and $f'(1) = \dfrac{1}{2}$. So the quation of the tangent line to $f(x)$ at $x = 1$ is

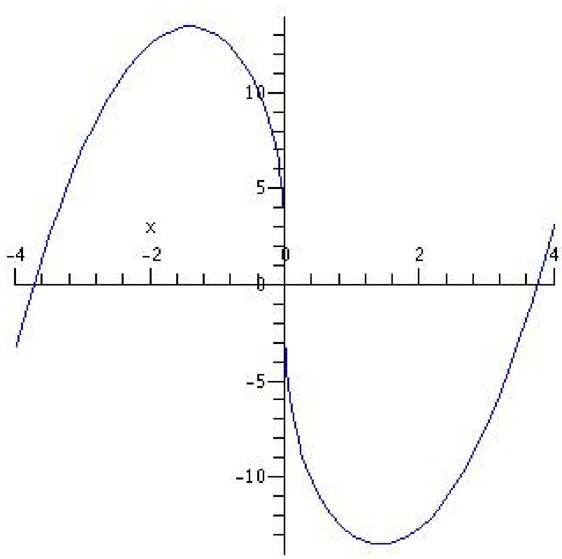
$$\frac{y - f(1)}{x - 1} = f'(1) \Leftrightarrow \frac{y - \pi/4}{x - 1} = \frac{1}{2} \Leftrightarrow y = \frac{1}{2}(x-1) + \frac{\pi}{4} = \frac{1}{2}x + \frac{\pi}{4} - \frac{1}{2}.$$

(c) [4 marks] all the inflection points on the graph of $y = x^{7/3} - 14\, x^{1/3}$

Solution: $f'(x) = \dfrac{7}{3}x^{4/3} - \dfrac{14}{3}x^{-2/3}$; $f''(x) = \dfrac{28}{9}x^{1/3} + \dfrac{28}{9}x^{-5/3} = \dfrac{28}{9}\left(\dfrac{x^2+1}{x^{5/3}}\right)$. So

$$f''(x) > 0 \Leftrightarrow x > 0 \text{ and } f''(x) < 0 \Leftrightarrow x < 0$$

and the only inflection point of f is the point $(0, 0)$. **Note:** $f''(0)$ is not defined; any one who states that "$f''(0) = 0$, and consequently there is an inflection point at $x = 0$," will lose 2 marks! For interest, the graph is below:



2. [avg: 7.85/10] Find the following:

(a) [3 marks] $\displaystyle\int_0^{\pi/2} 3\cos x\sqrt{1+3\sin x}\,dx$

Solution: let $u = 1+3\sin x$. Then $du = 3\cos x$ and

$$\int_0^{\pi/2} 3\cos x\sqrt{1+3\sin x}\,dx = \int_1^4 \sqrt{u}\,du = \left[\frac{2u^{3/2}}{3}\right]_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

(b) [2 marks] $\displaystyle\int_{-\pi/2}^{\pi/2} e^{-x^2}\sin x\,dx$

Solution: observe that the integrand is an odd function, so

$$\int_{-\pi/2}^{\pi/2} e^{-x^2}\sin x\,dx = 0.$$

Note: it would be nigh impossible, and a total waste of time, to try and find $\displaystyle\int e^{-x^2}\sin x\,dx$.

(c) [5 marks] $\displaystyle\int_0^{\pi} x^2\cos x\,dx$

Solution: integrate by parts. Start with $u = x^2, dv = \cos x\,dx$. Then $du = 2x\,dx,\ v = \sin x$. So

$$\begin{aligned}
\int x^2\cos x\,dx = \int u\,dv &= uv - \int v\,du \\
&= x^2\sin x - 2\int x\sin x\,dx \\
\text{(now let } s = x, dt = \sin x\,dx\text{)} \quad &= -x^2\sin x - 2\left(st - \int t\,ds\right) \\
&= x^2\sin x + 2x\cos x - 2\int \cos x\,dx \\
&= x^2\sin x + 2x\cos x - 2\sin x + C
\end{aligned}$$

and

$$\int_0^{\pi} x^2\cos x\,dx = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi} = -2\pi$$

3. [avg: 9.08/10] Let $v = t^2 - 5t + 6$ be the velocity of a particle at time t, for $0 \le t \le 3$. Find:

(a) [4 marks] the average velocity of the particle.

Solution: this is a direct application of the average formula. The average velocity is

$$\frac{1}{3-0}\int_0^3 v\,dt = \frac{1}{3}\int_0^3 (t^2 - 5t + 6)\,dt = \frac{1}{3}\left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^3 = \frac{3}{2}$$

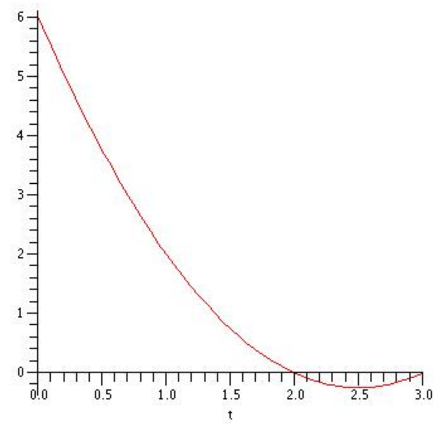
(b) [6 marks] the average speed of the particle.

Solution: the average speed is given by

$$\frac{1}{3}\int_0^3 |v|\,dt.$$

Since v changes signs on the interval $[0,3]$ we have to calculate two separate integrals:

$$\frac{1}{3}\int_0^3 |v|\,dt = \frac{1}{3}\int_0^2 v\,dt + \frac{1}{3}\int_2^3 (-v)\,dt$$



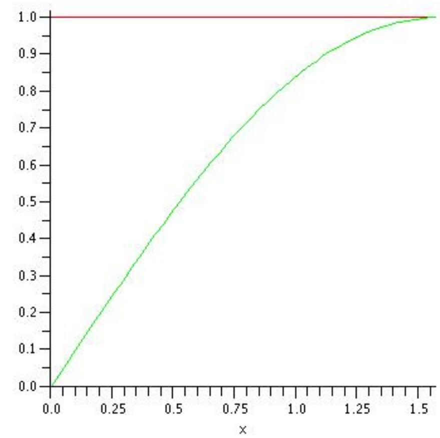
So the average speed is

$$\begin{aligned}
\frac{1}{3}\int_0^3 |v|\,dt &= \frac{1}{3}\int_0^2 (t^2 - 5t + 6)\,dt - \frac{1}{3}\int_2^3 (t^2 - 5t + 6)\,dt \\
&= \frac{1}{3}\left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^2 - \frac{1}{3}\left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_2^3 \\
&= \frac{14}{9} - \frac{1}{3}\left(-\frac{1}{6}\right) \\
&= \frac{14}{9} + \frac{1}{18} = \frac{29}{18}
\end{aligned}$$

4. [avg: 7.82/10] Let V be the volume of the solid obtained by revolving the region bounded by $y = \sin x$ and $y = 1$, for $0 \le x \le \pi/2$, around the y-axis.

(a) [3 marks] Use the method of shells to express the value of V as one integral with respect to x.

Solution: $V = \displaystyle\int_0^{\pi/2} 2\pi x(1 - \sin x)\, dx$



(b) [3 marks] Use the method of discs to express the value of V as one integral with respect to y.

Solution: $V = \displaystyle\int_0^1 \pi \left(\sin^{-1} y\right)^2 dy$

(c) [4 marks] Find the value of V.

Solution: either integral requires parts, but the easiest integral to evaluate is the one from (a):

$$\begin{aligned}
V = \int_0^{\pi/2} 2\pi x(1-\sin x)\, dx &= \pi \int_0^{\pi/2} 2x\, dx - 2\pi \int_0^{\pi/2} x \sin x\, dx \\
(\text{let } u = x, dv = \sin x) &= \pi \left[x^2\right]_0^{\pi/2} - 2\pi \left[-x\cos x\right]_0^{\pi/2} - 2\pi \int_0^{\pi/2} \cos x\, dx \\
&= \frac{\pi^3}{4} - 0 - 2\pi \left[\sin x\right]_0^{\pi/2} \\
&= \frac{\pi^3}{4} - 2\pi \text{ or } \pi\left(\frac{\pi^2 - 8}{4}\right)
\end{aligned}$$

Alternate Solution: for the integral from (b), you can use parts, twice. Start with $u = (\sin^{-1} y)^2,\ dv = dy$. Then $du = \dfrac{2\sin^{-1} y\, dy}{\sqrt{1-y^2}}, v = y$ and

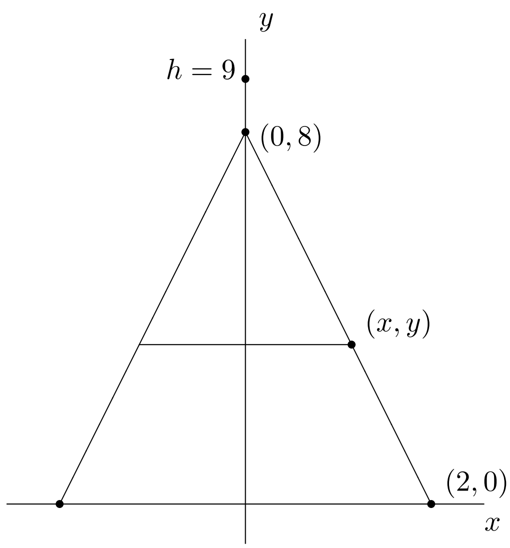
$$\begin{aligned}
\int \left(\sin^{-1} y\right)^2 dy &= y(\sin^{-1} y)^2 - \int \frac{2y \sin^{-1} y\, dy}{\sqrt{1-y^2}} \\
\left(\text{now let } s = \sin^{-1} y, dt = \frac{2y\, dy}{\sqrt{1-y^2}}\right) &= y(\sin^{-1} y)^2 - \left(st - \int t\, ds\right) \\
&= y(\sin^{-1} y)^2 - \left(-2\sqrt{1-y^2}\sin^{-1} y + \int 2\, dy\right) \\
&= y(\sin^{-1} y)^2 + 2\sqrt{1-y^2}\sin^{-1} y - 2y + C \\
\Rightarrow V &= \pi \left[y(\sin^{-1} y)^2 + 2\sqrt{1-y^2}\sin^{-1} y - 2y\right]_0^1 = \frac{\pi^3}{4} - 2\pi,
\end{aligned}$$

as before.

(See Page 10 for an alternate solution.)

5. [avg: 8.31/10] A tank is full of water. The tank is a square-based pyramid oriented with the tip of the pyramid pointing upward. Each side of the base of the pyramid is 4 m long, and the pyramid is 8 m tall. How much work is required to empty the tank by pumping all the water up to a height 1 m above the top of the tank? (Assume the density of water is ρ and that the acceleration due to gravity is g; leave your answer in terms of ρ and g.)

Solution: consider a side view. Let the tip of the pyramid be on the y-axis; let the base of the pyramid be on the x=axis.



Consider a horizontal cross-section at height y. See figure. The point (x, y) satisfies the equation

$$\frac{y}{8} + \frac{x}{2} = 1.$$

Thus

$$x = 2\left(1 - \frac{y}{8}\right) = \frac{1}{4}(8 - y)$$

and the cross-sectional area of the pyramid at height y is

$$A(y) = (2x)^2 = \frac{1}{4}(8 - y)^2.$$

Let W be the work required to empty the tank by pumping all the water up to a height 1 m above the top of the tank. Then

$$W = \int_0^8 \rho\, g\, A(y)(9 - y)\, dy.$$

The rest is calculation:

$$\begin{aligned}
W &= \int_0^8 \rho\, g\, A(y)(9-y)\, dy \\
&= \frac{\rho\, g}{4}\int_0^8 (8-y)^2(9-y)\, dy \\
(\text{let } u = 8 - y) \quad &= \frac{\rho\, g}{4}\int_8^0 u^2(u+1)\,(-du) \\
&= \frac{\rho\, g}{4}\int_0^8 (u^3 + u^2)\, du \\
&= \frac{\rho\, g}{4}\left[\frac{u^4}{4} + \frac{u^3}{3}\right]_0^8 = \frac{\rho\, g}{4}\left(\frac{8^4}{4} + \frac{8^3}{3}\right) \\
&= \frac{8^3 \rho\, g}{4}\left(2 + \frac{1}{3}\right) = 128\, \rho\, g\left(\frac{7}{3}\right) = \frac{896\, \rho\, g}{3}
\end{aligned}$$

Note: you could also use similar triangles to find $\dfrac{2x}{8-y} = \dfrac{4}{8} \Leftrightarrow 2x = \dfrac{8-y}{2}$.

6. [avg: 8.03/10] Consider the curve with equation $y = \dfrac{x^3}{12} + \dfrac{1}{x}$ for $1 \le x \le 4$.

(a) [5 marks] Find the length of the curve.

Solution: you need $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ for both parts of this question, so calculate it very carefully!

$$\begin{aligned}
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2} \\
&= \sqrt{1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}} \\
&= \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}} \\
&= \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2} \\
&= \frac{x^2}{4} + \frac{1}{x^2}, \text{ since } x^2 > 0.
\end{aligned}$$

Then the length of the curve is

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \frac{64}{12} - \frac{1}{4} - \frac{1}{12} + 1 = 6$$

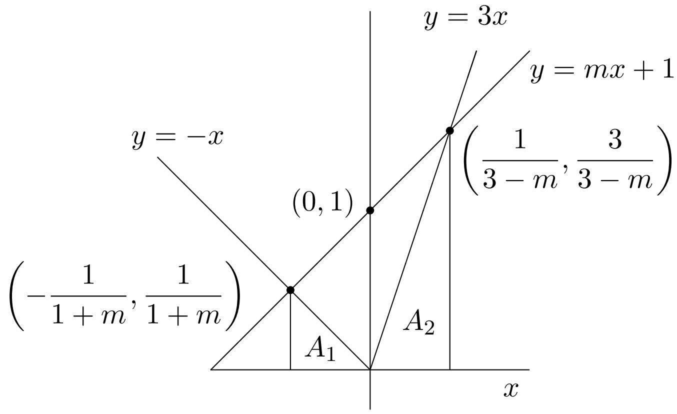
(b) [5 marks] Find the area of the surface generated by revolving the curve about the y-axis.

Solution: since we are revolving around the y-axis, the surface area is given by

$$\begin{aligned}
S &= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_1^4 2\pi x \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx \\
&= \frac{\pi}{2} \int_1^4 x^3\, dx + 2\pi \int_1^4 \frac{1}{x}\, dx \\
&= \frac{\pi}{2} \left[\frac{x^4}{4}\right]_1^4 + 2\pi \left[\ln x\right]_1^4 \\
&= 32\pi - \frac{\pi}{8} + 2\pi \ln 4 \\
&= \frac{255\,\pi}{8} + \pi \ln 16 \text{ or } \pi\left(\frac{255}{8} + \ln 16\right)
\end{aligned}$$

7. [avg: 5.09/10] Let A be the area of the region bounded by the graphs of $f(x) = x + 2|x|$ and $g(x) = m\,x + 1$. Find the value of m that minimizes the value of A.

Solution: first of all, to simplify $f(x)$ you need to take cases: $f(x) = \begin{cases} 3x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.



Next, you need to find the intersection points of $f(x)$ and $g(x)$:

$$\text{for } x < 0 : -x = mx + 1 \Rightarrow x = -\frac{1}{1+m};$$

$$\text{for } x > 0 : 3x = mx + 1 \Rightarrow x = \frac{1}{3-m}.$$

Let A_1 be the area of the triangle under the line $y = -x$; let A_2 be the area of the triangle under the line $y = 3x$. See figure to the left.

Then

$$\begin{aligned} A &= \int_{-1/(m+1)}^{1/(3-m)} g(x)\,dx - A_1 - A_2 \\ &= \left[\frac{m}{2}x^2 + x\right]_{-1/(m+1)}^{1/(3-m)} - \frac{1}{2(1+m)^2} - \frac{3}{2(3-m)^2} \\ &= \frac{m}{2(3-m)^2} + \frac{1}{3-m} - \frac{m}{2(1+m)^2} + \frac{1}{1+m} - \frac{1}{2(1+m)^2} - \frac{3}{2(3-m)^2} \\ &= \frac{1}{2(3-m)} + \frac{1}{2(m+1)} = \frac{2}{(3-m)(1+m)} \end{aligned}$$

$\boxed{\text{The problem is to minimize the value of } A \text{ for } -1 < m < 3.}$[1] Find the first and second derivatives:

$$\frac{dA}{dm} = \frac{1}{2(3-m)^2} - \frac{1}{2(m+1)^2}; \quad \frac{d^2A}{dm^2} = \frac{1}{(3-m)^3} + \frac{1}{(m+1)^3}.$$

Then

$$\frac{dA}{dm} = 0 \Rightarrow (3-m)^2 = (m+1)^2 \Rightarrow 9 - 6m + m^2 = m^2 + 2m + 1 \Rightarrow 8 = 8m \Rightarrow m = 1;$$

and at $m = 1$

$$\frac{d^2A}{dm^2} = \frac{1}{4} > 0.$$

So the value of A is minimized for $m = 1$. (See Page 10 for an alternate solution.)

[1] Note that if $m \leq -1$ or $m \geq 3$ then the area between the graphs of $f(x)$ and $g(x)$ is unbounded. Indeed, since $A \to \infty$ as $m \to -1^+$ or as $m \to 3^-$, A must have a minimum value on the interval $(-1, 3)$, and it must occur at a critical point. This observation can replace the second derivative test to confirm the minimum at $m = 1$.

8. [avg: 4.28/10] Gauss's error function, erf, is defined for all x by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. The first four values of $\text{erf}(n)$, for $n = 1, 2, 3, 4$ are:

$\text{erf}(1) \approx 0.842701,\ \text{erf}(2) \approx 0.995322,\ \text{erf}(3) \approx 0.999978,\ \text{erf}(4) \approx 0.999999.$

(a) [2 marks] Find the value of $\int_1^2 e^{-t^2}\, dt$ correct to four decimal places.

Solution:

$$\int_1^2 e^{-t^2}\, dt = \int_0^2 e^{-t^2}\, dt - \int_0^1 e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2} (\text{erf}(2) - \text{erf}(1)) \approx 0.135257\ldots$$

(b) [2 marks] Find $\text{erf}\,'(x)$ and determine for which values of x the error function is increasing.

Solution:

$$\text{erf}\,'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} > 0,$$

for all x. So the error function is increasing for all x.

(c) [2 marks] Show that $\text{erf}(-x) = -\text{erf}(x)$.

Solution:

$$\begin{aligned} \text{erf}(-x) &= \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt \\ (\text{let } u = -t) &= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\, du \\ &= -\text{erf}(x) \end{aligned}$$

(d) [4 marks] Find the value of $\int_0^1 \text{erf}(x)\, dx$ correct to four decimal places.

Solution: use integration by parts. Let $u = \text{erf}(x), dv = dx$. Then $v = x, du = \frac{2}{\sqrt{\pi}} e^{-x^2} dx$, and

$$\begin{aligned} \int_0^1 \text{erf}(x)\, dx &= [x\ \text{erf}(x)]_0^1 - \frac{2}{\sqrt{\pi}} \int_0^1 x e^{-x^2}\, dx \\ (\text{let } w = -x^2) &= \text{erf}(1) - 0 + \frac{1}{\sqrt{\pi}} \int_0^{-1} e^{w}\, dw \\ &= \text{erf}(1) + \frac{1}{\sqrt{\pi}} \left(\frac{1}{e} - 1 \right) \\ &\approx 0.48606\ldots \end{aligned}$$

Alternate Solutions:

Question 4: let $\theta = \sin^{-1} y$. Then $d\theta = \dfrac{dy}{\sqrt{1-\sin^2\theta}} = \dfrac{dy}{\cos\theta} \Leftrightarrow dy = \cos\theta\, d\theta$ and

$$\begin{aligned} V &= \int_0^1 \pi \left(\sin^{-1} y\right)^2 \, dy \\ &= \pi \int_0^{\pi/2} \theta^2 \cos\theta \, d\theta \\ \text{(from Question 2(c))} &= \pi \left[\theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta\right]_0^{\pi/2} \\ &= \pi\left(\frac{\pi^2}{4} - 2\right), \end{aligned}$$

as before.

Question 5: alternate calculations

$$\begin{aligned} W &= \rho\, g \int_0^8 16\left(1 - \frac{y}{8}\right)^2 (9-y)\, dy \\ &= \rho\, g \int_0^8 \left(16 - 4y + \frac{y^2}{4}\right)(9-y)\, dy \\ &= \rho\, g \int_0^8 \left(144 - 52y + \frac{25y^2}{4} - \frac{y^3}{4}\right) dy \\ &= \rho\, g \left[144y - 26y^2 + \frac{25y^3}{12} - \frac{y^4}{16}\right]_0^8 \\ &= \rho\, g\left(\frac{896}{3}\right), \end{aligned}$$

as before.

Question 7: since A is the area of a triangle, you could use *vectors* and find A using the *cross-product:*

$$\begin{aligned} A &= \frac{1}{2}\left\| \begin{bmatrix} 1/(3-m) \\ 3/(3-m) \\ 0 \end{bmatrix} \times \begin{bmatrix} -1/(1+m) \\ 1/(m+1) \\ 0 \end{bmatrix} \right\| \\ &= \frac{1}{2}\left(\frac{1}{(3-m)(m+1)} + \frac{3}{(3-m)(1+m)}\right) \\ &= \frac{2}{(3-m)(m+1)} \end{aligned}$$

and then find the critical point of A as before. OR, in the same vein, you can avoid integration by using the formula for the area of a trapezoid to get the value of the area under $g(x)$ from one intersection point to the other.

Alternate Solutions:

Question 8(c): consider the function $g(x) = \text{erf}(-x) + \text{erf}(x)$. By the chain rule and the fundamental theorem of calculus, we have

$$g'(x) = -\text{erf}'(-x) + \text{erf}'(x) = -\frac{2}{\sqrt{\pi}}e^{-(-x)^2} + \frac{2}{\sqrt{\pi}}e^{-x^2} = 0,$$

so $g(x)$ is constant. Since $g(0) = 0$, we must have $g(x) = 0$ for all x, which implies $\text{erf}(-x) = -\text{erf}(x)$.

Question 8(d): work with "double integrals," which is beyond the scope of this course, but many students tried it. It can work if you know what you are doing:

$$\int_0^1 \text{erf}(x)\,dx = \frac{2}{\sqrt{\pi}}\int_0^1\int_0^x e^{-t^2}\,dtdx.$$

Interchanging the order of integration, we have

$$\begin{aligned}
\frac{2}{\sqrt{\pi}}\int_0^1\int_0^x e^{-t^2}\,dtdx &= \frac{2}{\sqrt{\pi}}\int_0^1\int_t^1 e^{-t^2}\,dxdt \\
&= \frac{2}{\sqrt{\pi}}\int_0^1 (1-t)e^{-t^2}\,dt \\
&= \text{erf}(1) - \frac{2}{\sqrt{\pi}}\int_0^1 te^{-t^2}\,dt \\
&= \text{erf}(1) + \frac{1}{\sqrt{\pi}}(e^{-1} - 1) \\
&\approx 0.4861.
\end{aligned}$$

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

The end.